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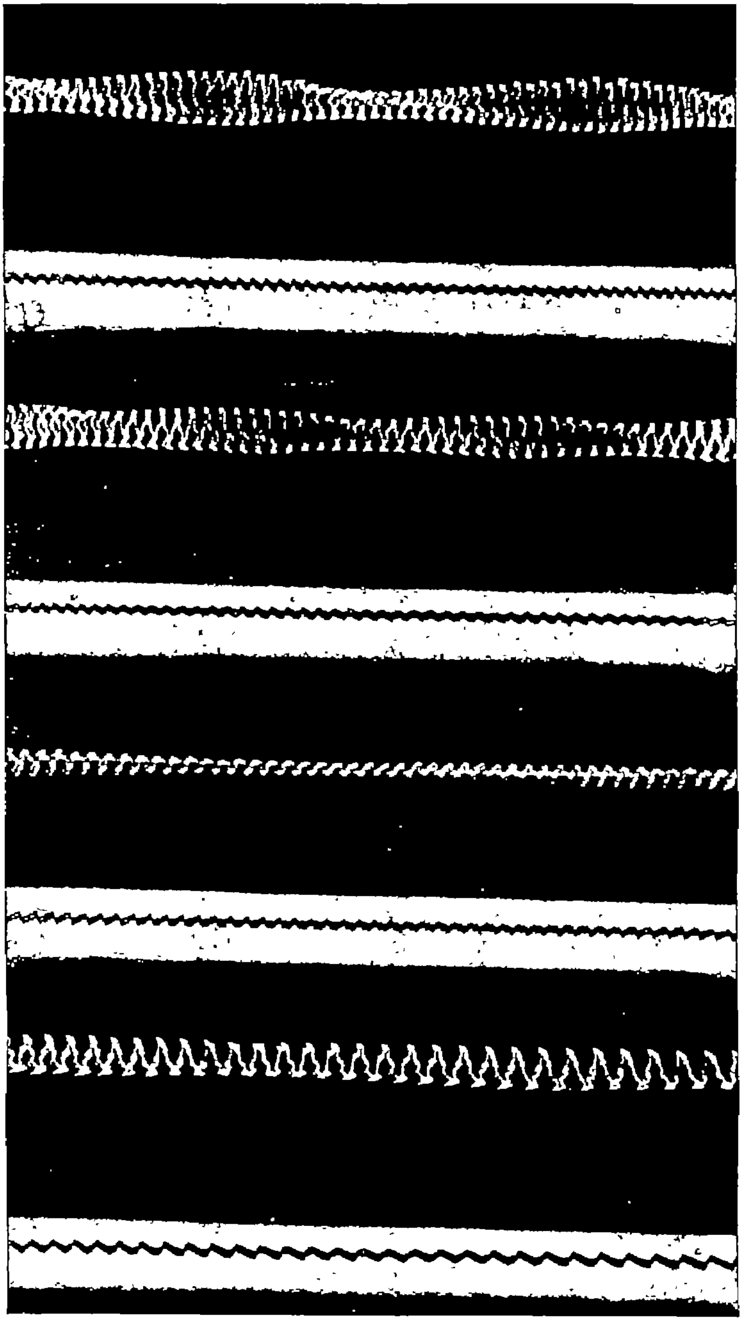
PLATE I.



Simultaneous Vibration-Curves of the Bridge and D-String of a Violoncello, showing Cyclical changes at frequencies other than that of the "Wolf-note."

The Theory of the Cyclical Vibrations of a Bowed String.

BY

C. V. RAMAN.

(PLATE I)

Introduction.

Ordinarily, the motion of every part of a stringed instrument excited by bowing is a periodic vibration capable of being represented by a simple Fourier series. The motion of the string and that of the bridge, belly and air, are all strictly periodic. Recently, in investigating a phenomenon on the 'cello known as the "wolf-note", which is observed when the G-string or D-string is bowed at a pitch equal or nearly equal to that of the strongest resonance of the instrument, G. W. White* found that the motion of the belly was of large amplitude and, instead of being periodic, was cyclical in character. White considered the phenomenon to be due to the beats which arise when the pitch impressed on the system approaches the natural pitch of the resonator. This view, though it seems a plausible one at first sight, is open to objection, and on careful examination is found to be inconsistent with the observed facts. Perhaps the most effective criticism of the suggestion put forward by White is to be found in the fact first noticed by me that cyclical changes in the vibration of the belly can be obtained by bowing the G-string or D-string of a 'cello, rather near the bridge and with a suitable pressure, at *any* pitch lying between, say, 210 and 370 vibrations per second. (The pitch of the wolf-note is about 176 vibrations per second, and there is another fairly strong point of resonance at 360 vibrations per second). This fact alone is sufficient to dispose of any suggestion that the cyclical changes require for their production any close approximation between the natural frequency of the resonator and the impressed frequency of vibration.

From fundamental dynamical principles, it is evident that, if in any particular case the motion of the string is strictly periodic, the maintained vibration of every other part of the instrument must also be strictly periodic. It may be inferred, therefore, that when the motion of the belly is not simply periodic but is of a cyclical character, the vibrations of the string must also be of a cyclical character, and that the changes in the latter must precede (not follow) the corresponding changes in the former. This indication of theory can easily be verified experimentally, as shown in the communication by me published in 'Nature'.* As mentioned above, such cyclical changes occur not only at the wolf-note pitch, but also, under suitable conditions, at other frequencies of vibration, and any theory of the phenomenon must be capable of explaining this fact.

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* G. W. White, *Proceedings of the Cambridge Philosophical Society, June 1915.*

Modus Operandi of the Bow.

It is in the mode of action of the bow that we must look for an explanation of the cyclical changes in the character of vibration of the string that occur under certain conditions. The system on which the bow acts consists of a stretched cord, one end of which is practically fixed, and the other end of which passes over the bridge, and is therefore capable of yielding under the periodic transverse and longitudinal components of the tension. Since the free periods of the string are modified by the yielding of the bridge, the problem of finding the motion maintained by the bow is by no means a simple one. As a preliminary to the discussion of the cyclical cases, we shall first briefly consider the general theory of the much more simple cases in which the vibration is of a periodic character.

The dissipation of energy through the communication of vibrations to the atmosphere as sound-waves is evidently the central feature of the problem. The maintenance of a steady, stable form of vibration of the string is only possible if there is an exact balance between these energy losses and the energy drawn from the bow, and if this balance is also not liable to be upset by any slight alteration in the pressure of the bow. If we assume that the pressure with which the bow is applied is sufficiently large, such energy-balance is only possible when the bowed point is carried forward by the bow with its own velocity for a considerable fraction of the period, and during the other part or parts of the period of vibration, slides down with a velocity which is not necessarily uniform or constant. For, if the relative velocity at the bowed point did not thus become zero during a considerable part of the period, the maintaining forces (due to the difference in the friction acting on the bowed point in the forward and backward motions) would be far in excess of those required to maintain the motion; whereas, by the relative velocity actually becoming zero, the frictional force in such stages falls below the maximum statical value and can thus automatically adjust itself so as to secure the requisite balance of energy supply and loss. Mathematically expressed, the relation may be written as

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* "On the Wolf-note of the Violin and 'Cello," *Nature*, 29th June, 1916, page 362-363. A fuller discussion of the phenomena observed at the wolf-note pitch will shortly appear in the *"Philosophical Magazine."*

$$F\left(P, v-v_B\right)=P_o+\sum_{n=1}^{n=\infty} k_n B_n \frac{Sin\left(\frac{2n\pi t}{T}+e_n+e'_n\right)}{Sin\ \frac{n\pi x_o}{l}},$$

where P is the pressure of the bow, v_B its velocity, v the velocity of the bowed point, and $F\left(P, v-v_B\right)$ is the frictional force which is a function of the pressure and relative velocity. On the right-hand side, P_o is a constant, $B_n\ Sin\left(\frac{2n\pi t}{T}+e_n\right)$ represents the n^{th} harmonic component in the vibration of the string, x_o is the distance of the bowed point from the fixed end, and k_n, e_n' are also constants. One side of the equation thus represents the force exerted by the bow, and the other side, the resultant of the forces required to maintain the vibration in the presence of dissipative forces.

From this result indicating the general nature of the motion at the bowed point, the next step is to deduce by kinematical analysis, the characteristics of the various possible modes of vibration, and then, by summation of the series for the maintaining force, to find which of the types indicated by the kinematical analysis is dynamically possible for any given pressure or velocity of bowing. In two notes already published in this Bulletin,* I have already indicated the most convenient method of kinematical analysis. For our present purpose, it will be sufficient to remark that when the bow is applied at any point considerably removed from an end of the string, a considerable variety of possible modes of vibration is indicated. When however, the bow is applied near an end of the string, the range of possibilies becomes greatly restricted. In this case, if the motion at the bowed point be approximately representable by a two-step zig-zag vibration curve, the motion of every other point on the string is also approximately of a similar kind, as found by Helmholtz,† and the motion is uniquely determined, the only alternatives being types in which the vibration-curve at the bowed point is a four-step or a six-step zig-zag etc.

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* Vol. IV, pages 1-4 and Vol. V, pages 5-8.

† See also the paper by myself and another on 'Discontinuous Wave motion' *Philosophical Magazine*, January, 1916.

Conditions under which the motion is cyclical.

The mechanical theory of the periodic types of vibration thus indicates that when the bow is applied close to an end of the string, and its pressure is insufficient to maintain the well-known simple type of vibration described by Helmholtz, the motion at the bowed point must alter to a type in which it is a four-step zig-zag or a six-step zig-zag, etc., the fundamental becoming feeble and even falling out altogether. The case is quite different if the bow be applied at a point sufficiently removed from the end ; the motion at the bowed point may then alter to a very considerable extent so as to adjust itself to a change in the pressure of the bow, and yet remain approximately a two-step zig-zag. In view of this, it is a significant experimental fact that cyclical changes of vibration-form occur only when the bow is applied with suitable pressure and velocity near an end of the string, and not when it is applied at a point sufficiently removed therefrom. We may reasonably infer that cyclical changes occur only when the pressure of bowing and other conditions are such that no one steady, stable form of vibration can be maintained.

In order to fix our ideas, we may consider a specific case in which the quantity k_1 is very large, in comparison with k_2, k_3, etc. Since k_1 relates to the fundamental component, it is evident from the expression given above, that the pressure of the bow must be very considerable in order that the ordinary type of vibration in which the fundamental is dominant might be maintained. The critical value of the pressure necessary is evidently $k_1 B_1 (1 - Sin\ \epsilon_1') \big/ Sin\ \frac{\pi x_0}{l}\ (\mu - \mu_A)$, where μ and μ_A are the coefficients of statical and dynamical friction, respectively. When the pressure is smaller than this critical value, the mode of vibration in which the fundamental is dominant becomes impossible. The various alternatives which then arise are, (*a*) the fundamental may fall out altogether, in which case the string would vibrate in two segments in the usual Helmholtzian mode : or (*b*), intermediate forms may arise. The contingency in (*a*) may arise, and generally does arise, if the pressure of the bow be reduced to a value not much in excess of the critical pressure for the type of vibration in two segments. The string would then settle down to a steady state of vibration. The intermediate forms of vibration referred to in (*b*), would only be possible when the pressure of the bow is less than the minimum necessary te ensure a steady vibration with dominant fundamental, but still much in excess of that required for the contingency in (*a*). It now remains to be seen why, in the conditions which exist when cyclical forms of vibration are set up, no steady state of vibration is possible.

The kinematical theory of the intermediate forms referred to in the preceding para is readily worked out, at any rate to a close approximation. They may be represented sufficiently closely by a mode of vibration in which two discontinuous changes of velocity travel along a string whose velocity-diagram consists of parallel straight lines. The discontinuous changes are, in general, unequal. The vibration-curve at the bowed point would, in general, be a four-step zig-zag in which the two ascending lines are straight and parallel, and the two descending lines are straight but not necessarily parallel to one another. In order that a steady vibration in such a mode might be possible, it is necessary that the force necessary to maintain it should be less than the statical friction in the two stages of ascent, and exactly equal to the values of the dynamical friction in the two stages of descent. As we assumed that k_1 is much larger than k_2, k_3 etc, the force required to maintain the fundamental is far in excess of those required to maintain the second, third harmonics, etc. Since a large value of k_1 is generally the result of the fundamental being strongly re-inforced by the resonance of the instrument, the phase constant e_1' would also be considerably different from the phase constants e_2' , e_3' etc. It is readily shown that the resultant obtained by superposing these forces would not, in general, even approximately satisfy the condition for steady maintenance stated above ; even if it did approximately satisfy the condition for some particular value of the bowing pressure, it would not do so for other values of the pressure, and the motion would not therefore be stable. It is thus found that for the values of the bowing pressure contemplated, none of the intermediate forms of vibration is capable of being steadily maintained. Since a finite interval of time is required for any change in the amplitude or form of vibration, either of the string or of the rest of the instrument, we see that the conditions of the case discussed are favourable for the production of cyclical forms of vibration.

Similarly when both k_1 and k_2 are very large compared with k_3, k_4, etc., or when k_2 by itself is large compared with the constants k_1, k_3, etc., and the bow is applied near an end of the string, cyclical forms of vibration are possible.

The cases in which k_3 or k_4 etc., are large compared with k_1, k_2 etc., are not of special interest. For, the amplitudes of the third and higher harmonics are small compared with that of the fundamental in the motion ordinarily maintained by bowing near an end of the string, and their influence is therefore too small to affect the possibility of a steady state of vibration.

Nature of the Cyclical Process.

The main interest of the problem is in finding the character of the motion of the string at successive stages of the cycle, its relation to the coresponding changes in the vibration of the bridge and belly, the factors that determine these changes, and the number of periods of vibration comprised in each cycle. A rigorous mathematical formulation of all the conditions, though not impossible, would evidently be too complicated to be practically useful. The main results may however, be arrived at from broad theoretical considerations.

In the initial stage, when the bow acts upon the string *with a moderate pressure,* it would evidently be capable of setting up a form of vibration which, later on, would fail to be maintained on account of the insufficient pressure of the bow when the belly attains its maximum vibration. The change in the form of vibration of the string thus caused would, in its turn, result in a falling off of the vibration of the belly, but on account of the inertia of the belly, this change would naturally lag behind that of the string to a considerable extent. By the time the belly vibration reaches its minimum, the string would have already attained a form of vibration which, with the reduced amplitude of the bridge-vibrations, takes up considerably less energy than the bow is capable of yielding. As a result, the string commences to regain its original form of vibration, and this is subsequently followed by a revival of the vibrations of the belly. The cycle then repeats itself indefinitely.

While the description of the cyclical process given above holds good generally, individual cases differ very considerably in detail. If k_1 is large, the largest changes in amplitude occur in the fundamental component of the vibration of the string, and the vibration—curve of a point near an end of it passes by successive stages from the simple two-step to a four-step zig-zag (in which the fundamental is more or less feeble), and then back again to a two-step zig-zag. When k_2 is large, the principal changes are in the amplitude of the second harmonic, and the case is therefore somewhat more complicated than the first.

When both k_1 and k_2 are large, we may have more than one kind of cyclical change possible.

As remarked above, the changes in the vibration of the belly lag behind those of the string by a considerable interval. Experiment shows this interval in a good many cases to be approximately quarter of a cycle. The total number of periods comprised in each cycle is evidently determined by the pressure of the bow, the frictional constants and the quantities k_1, k_2 etc. When k_1 or k_2 is large, the period of the cycle is practically a function of k_1 or k_2 as the case may be. It is then approximately twice the interval of time in which the vibrations of the belly would decrease from the largest to the smallest amplitude actually observed in the cycle, if the belly were first excited by steadily bowing the string and the bow were then suddenly removed.

Some Experimental Results.

That the dissipation of energy is the controlling factor in determining the phenomena discussed in this paper, may be very prettily illustrated by loading the bridge of a 'cello with a brass clamp weighing about 44 grammes. Instead of getting two resonance-frequencies at 176 and 360 vibrations per sec., as is the case when there is no load on the bridge, we then get four resonance-points, whose frequencies are 100, 137, 184 and 233.

At the first and third of these frequencies, a very considerable vibration of the bridge and belly may be set up by bowing the string, but on removing the bow the vibration dies away quite slowly, showing that the rate of dissipation of energy is quite small at these frequencies. Cyclical effects are hardly noticeable at these frequencies, even with carefully-adjusted bowing pressure. At the two frequencies 136 and 237, however, the damping is much more marked, (though it is still considerably less than at the two corresponding frequencies 176 and 360 when there is no load on the bridge), and cyclical effects are obtained fairly easily.

Plate II shows the cyclical vibrations of the bridge and D—string of a 'cello which had no load fixed on the bridge. The first two pictures were taken with different bowing pressures, at the second resonance point with a frequency of 360 vibration per second. It will be noticed that in the second picture, the cycle is somewhat more rapid than in the first, and the difference between the maximum and minimum aplitudes of vibration of the bridge (as shown by the white curve on the dark ground) is less in the second than in the first. The dark curve on the white ground shows the cyclical vibrations of the string, observed at a point whose distance from the bridge, though small, was still a considerable fraction of the vibrating length of the string. The cyclical changes, though visible, are not very marked. The third and fourth pictures represent cyclical effects obtained with two different bowing pressures and velocities at a frequency of 240 vibrations per second. The difference in the amplitude of vibration of the belly, and in the character of the changes in the vibration-form of the the string in the two cases is very marked. In all the curves, the axis of time runs from right to left

On the motion of a perfectly elastic particle inside a given plane area nnder no external forces.

By

NALINIMOHAN BASU.

The present paper is the first instalment of the results of my investigation of the conditions under which a perfectly elastic particle must be projected inside a given plane area in order that, after a finite number of impacts from the boundary, it may retrace its path. The simplest case of this problem, *viz.* that in which the given area is a square, has been recently studied by two well-known Hungarian mathematicians, Messrs. König and Szücs.*

I wish to express my indebtedness to Dr. Ganesh Prasad at whose suggestion I took up, and under whose guidance I carried on, the investigation of this problem.

Case I. The Equilateral Triangle.

1. Let the equilateral triangle ABC be the given area and let O be its centroid. Suppose that the particle is projected from the point P in the direction PQ. Now if we imagine plane mirrors to be placed on the sides AB, BC, CA perpendicular to the plane of the triangle ABC, then ABC' may be regarded as an image of the triangle ABC and P_1 that of P. There will be two other images corresponding to the sides BC and CA and in each there will be an image of P. In a similar way, we can form images of these triangles, then their images, and so on, and the whole plane will be divided into an infinite number of image-triangles which we shall call *images of the triangle ABC.* Again in each of these images of ABC, there will be a point which is either an image of the point P or of one of its images; to avoid circumlocution, we shall call all these points *images of* P.

2. These points will all have motions similar to that of the particle in the triangle ABC. Hence if at any time the particle repasses its point of starting, all the images of this particle will repass their points of starting.

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* "Mouvement d'un point abandonné a l' intérieur d'un cube." (*Rendiconti del Circolo matematico di Palermo,* Vol. 36, 1913.)

3. Now some of the image-triangles will have their sides parallel, in the same sense, to the corresponding sides of the triangle ABC. These we shall call the *direct images*. and the first set of direct images, which lie symmetrically with respect to the triangle ABC, we shall call the *primary direct images;* and the corresponding images of any point we shall call the direct images, and the primary direct images, respectively, of that point. It will be found from a simple geometrical construction that there are two primary direct images of O on each of the three perpendiculars from it on the sides of the triangle ABC and that they lie at distances $\pm 2p$ from O, where p is the perpendicular distance of any angular point from the opposite side. Therefore, taking O as the origin, and the axes of x and y as in the figure on page 11, we find for the coordinates of the centroids of the six primary direct images of ABC the following:

$\left(2p\cos\frac{\pi}{6}, 2p\sin\frac{\pi}{6}\right)$, $\left(-2p\cos\frac{\pi}{6}, -2p\sin\frac{\pi}{6}\right)$, $\left(2p\cos\frac{5\pi}{6}, 2p\sin\frac{5\pi}{6}\right)$, $\left(-2p\cos\frac{5\pi}{6}, -2p\sin\frac{5\pi}{6}\right)$, $\left(2p\cos\frac{9\pi}{6}, 2p\sin\frac{9\pi}{6}\right)$, and $\left(-2p\cos\frac{9\pi}{6}, -2p\sin\frac{9\pi}{6}\right)$.

Again the direct images of these centroids are formed according to the same law and therefore all the direct images of ABC are obtained from the following coordinates of their centroids:—

$$x = 2p\left[m_1\cos\frac{\pi}{6} + m_2\cos\frac{5\pi}{6} + m_3\cos\frac{9\pi}{6}\right],$$

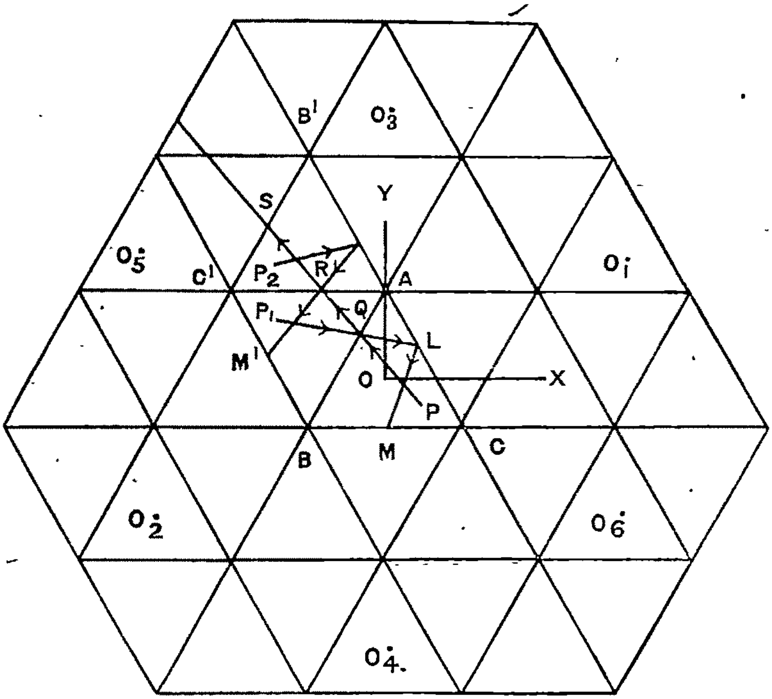
$$y = 2p\left[m_1\sin\frac{\pi}{6} + m_2\sin\frac{5\pi}{6} + m_3\sin\frac{9\pi}{6}\right],$$

where m_1, m_2, m_3 are any integers whatsoever, positive, negative or zero. Therefore, if (α, β) be the coordinates of P, the point of projection, the coordinates of its direct images are given by

$$x_1 = \alpha + 2p\left[m_1\cos\frac{\pi}{6} + m_2\cos\frac{5\pi}{6} + m_3\cos\frac{9\pi}{6}\right],$$

$$y_1 = \beta + 2p\left[m_1\sin\frac{\pi}{6} + m_2\sin\frac{5\pi}{6} + m_3\sin\frac{9\pi}{6}\right].$$

4. Now, as the particle impinges on the side A B at Q, its image in the triangle ABC′ simultaneously impinges on that side at the same point, and as the particle after rebound proceeds to describe the path Q L, the image in ABC′ proceeds to describe Q R, the portion of the line P Q intercepted by that triangle. When the particle reaches L, the image in ABC′ reaches R and the image in AB′C′ also reaches R;


B'
O'3
Y
S
O'5
C'
P2
R
A
O'1
P1
Q
L
M'
O
X
P
B
M
C
O'2
O'6
O'4

and as the particle describes the path LM, the image in ABC′ describes R M′ and that in AB′C′ describes R S, the portion of the line P Q intercepted by AB′C′. So we see that the line P Q is divided into an infinite number of segments by the image-triangles, and that each segment is described, in course of time, by the image in the triangle corresponding to that segment. If therefore the direction of projection is such that the line P Q passes through an image P_1 of P, the moving image in the corresponding triangle will, when it describes the corresponding segment of the line PQ, repass its initial position P_1, and so, all other images, and the particle itself, will repass their respective initial positions. Therefore the necessary and sufficient condition that the particle may, at any instant, repass its initial position is that P Q must pass through an image of P.

5. Now, as it is only a direct image whose motion at any instant is parallel, in the same sense, to that of the particle, it is evident that if the particle is to retrace its path, the line P Q must pass through a direct image. Therefore, if θ be the angle which the direction of projection makes with the x-axis, we must have, for the necessary and sufficient condition that the particle may retrace its path,

$$\tan\theta = \frac{y_1 - \beta}{x_1 - \alpha} = \frac{2p\left(m_1 \sin\frac{\pi}{6} + m_2 \sin\frac{5\pi}{6} + m_3 \sin\frac{9\pi}{6}\right)}{2p\left(m_1 \cos\frac{\pi}{6} + m_2 \cos\frac{5\pi}{6} + m_3 \cos\frac{9\pi}{6}\right)}$$

$$= \frac{m_1 + m_2 - 2m_3}{m_1 - m_2} \cdot \frac{1}{\sqrt{3}}$$

$$= \frac{\lambda}{\mu} \cdot \frac{1}{\sqrt{3}},$$

where λ and μ are any integers whatsoever, positive, negative or zero, for m_1, m_2 and m_3 may always be so determined that

$$\frac{m_1 + m_2 - 2m_3}{m_1 - m_2} = \frac{\lambda}{\mu}.$$

Case II. The Regular Hexagon.

6. Let the given area be a regular hexagon. In this case also it is evident that the whole plane will be divided into an infinite number of image-hexagons. It is also clear that there will be six primary direct images of the hexagon, and that their centroids lie one on each of the perpendiculars drawn from O, the centroid of the hexagon, upon the six sides, at distance $2p$ from O, where p is the

distance between any two parallel sides. Therefore the co-ordinates of the six primary direct images of O are $\left(2p\cos\frac{\pi}{6},\ 2p\sin\frac{\pi}{6}\right)$, $\left(-2p\cos\frac{\pi}{6},\ -2p\sin\frac{\pi}{6}\right)$, $\left(2p\cos\frac{3\pi}{6},\ 2p\sin\frac{3\pi}{6}\right)$, $\left(-2p\cos\frac{3\pi}{6},\ -2p\sin\frac{3\pi}{6}\right)$, $\left(2p\cos\frac{5\pi}{6},\ 2p\sin\frac{5\pi}{6}\right)$, and $\left(-2p\cos\frac{5\pi}{6},\ -2p\sin\frac{5\pi}{6}\right)$, the origin being O. Therefore the co-ordinates of all the direct images of O are given by

$$x = 2p\left[m_1\cos\frac{\pi}{6} + m_2\cos\frac{3\pi}{6} + m_3\cos\frac{5\pi}{6}\right],$$

$$y = 2p\left[m_1\sin\frac{\pi}{6} + m_2\sin\frac{3\pi}{6} + m_3\sin\frac{5\pi}{6}\right],$$

where m_1, m_2, m_3 are any integers, positive, negative or zero. Therefore if (α, β) be the point of projection, its direct images are given by $x_1 = \alpha + x$, $y_1 = \beta + y$, x and y being given by the above relation.

7. In this case also it is clear that the line of projection is divided into an infinite number of segments and that each segment in turn is described by an image. Therefore, as in the previous case, the necessary and sufficient condition that the particle may retrace its path is that the line of projection must pass through a direct image of the point of projection. Accordingly if θ be the inclination of the line of projection to the x— axis, we must have, for the required condition

$$\tan\theta = \frac{y_1-\beta}{x_1-\alpha}$$

$$= \frac{2p\left(m_1\sin\frac{\pi}{6} + m_2\sin\frac{3\pi}{6} + m_3\sin\frac{5\pi}{6}\right)}{2p\left(m_1\cos\frac{\pi}{6} + m_2\cos\frac{3\pi}{6} + m_3\cos\frac{5\pi}{6}\right)}$$

$$= \frac{m_1 + 2m_2 + m_3}{m_1 - m_3}\cdot\frac{1}{\sqrt{3}},$$

$$= \frac{\lambda}{\mu}\cdot\frac{1}{\sqrt{3}},$$

where λ and μ are any integers whatsoever, positive, negative or zero, for m_1, m_2 and m_3 may always be so determined that $\frac{m_1+2m_2+m_3}{m_1-m_3} = \frac{\lambda}{\mu}$.

8. In a subsequent communication I hope to give the necessary and sufficient condition for the general case in which the boundary is a regular n-agon. It is probable that the method of images will prove successful in the general case, although up to now, this case has baffled my attempts.

As regards the cases in which the boundary is curvilinear, the case of the circle admits of easy treatment. I hope to be able to obtain the necessary and sufficient condition for the oval next in order of complexity, *viz.* the ellipse. In the present state of analysis, it is evidently impossible to attempt, with any hope of success, the general case in which the given boundary is *any* oval.

Parametric Co-efficients in the Differential Geometry of Curves.

BY

SYAMADAS MUKHOPADHYAY.

VI—On Nul Parametric Co-efficients and Osculating Spherics to a Curve in an N-Space

1. Let x_r $(r=1, 2, ., n)$ be the n co-ordinates of a point, referred to n mutually orthogonal axes in an n-space, which may be called the point x. If x depend on a single variable parameter t the locus of x will be in general a curve in the n-space.

Suppose $\overset{(p)}{x}_r$ denotes the p^{th} derivative of x_r with respect to t. Then

$$\Sigma \overset{(p)}{x}_r \overset{(q)}{x}_r, \ (r=1, 2, \cdot, n)$$

has been called a parametric co-efficient of class I. It is symbolically indicated by (p, q), where p and q are positive integers. We will now introduce and interpret parametric co-efficients in which one or both of the indices p and q are zero. Such parametric co-efficients may be called *nul-parametric co-efficients.*

The parametric co-efficient (p, q) is evidently the limit of

$$\Sigma\left(X_r - x_r\right)^{(p)} \left(X_r - x_r\right)^{(q)}, \ (r=1, 2, \cdot, n)$$

when $X=x$, where X is a *variable* point on the curve in the neighbourhood of the *fixed* point x. We will define the nul parametric co-efficients (o, p) and (o, o) as the limits of

$$\Sigma\left(X_r - x_r\right)\left(X_r - x_r\right)^{(p)} \text{ and } \Sigma\left(X_r - x_r\right)\left(X_r - x_r\right)$$

respectively, when $X=x$.

As (o, p) and (o, o) are both constantly zero along the curve, their derivatives of any order with respect to t are also zero. We may however derive from each of them a set of functions by making the rules of differentiation of a parametric co-efficient when both indices are positive integers, also applicable to them.

The rules for differentiating (p, q) give, as we know,

$$(p, q)' = (p, q+1) + (p+1, q)$$

$$(p, q)^{(r)} = (p, q+r) + r\,(p+1, q+r-1)$$

$$+\frac{r(r-1)}{1\cdot 2}\,(p+2, q+r-2) + \ldots + (p+r, q)$$

We will define

$$(o, p)_1 = (o, p+1) + (1, p) = (1, p)$$

$$(o, p)_r = (o, p+r) + r\,(1, p+r-1) + \frac{r\,(r-1)}{1.2}\,(2, p+r-2)$$

$$+ \ldots + (r, p)$$

$$= r\,(1, p+r-1) + \frac{r\,(r-1)}{1.2}\,(2, p+r-2) + \ldots + (r, p)$$

and $(o, o)_1 = (0, 1) + (1, 0) = 0$

$$(o, o)_2 = (0, 2) + 2\,(1, 1) + (2, 0) = 2(1, 1)$$

$$(o, o)_3 = (0, 3) + 3\,(1, 2) + 3\,(2, 1) + (3, 0) = 6\,(1, 2)$$

……………………………………………………………………

$$(o, o)_r = (0, r) + r\,(1, r-1) + \frac{r\,(r-1)}{1{,}2}\,(2, r-2) + \ldots + (r, 0)$$

$$= 2\,(1, r-1) + \frac{2\,r(r-1)}{1,\,2}\,(2, r-2) + \ldots \qquad (i)$$

It is evident from the definitions of $(o, p)_r$ and $(o, o)_r$, that

$$(o, p)_r = \underset{X=x}{Lt}\left(\frac{d}{dt}\right)^r \Sigma\left(X_r - x_s\right)\left(X_s - x_s\right)^{(p)},\ (s=1, 2, ., n)$$

and $(o, o)_r = \underset{X=x}{Lt}\left(\frac{d}{dt}\right)^n \Sigma\left(X_s - x_s\right)^2,\ (s=1, 2, ., n)$

where X is a *variable* point in the neighbourhood of the *fixed* point x.

In Paper III we have defined $\Delta_r\,(p)$ and in Paper IV we have given a table of values of $\Delta_r\,(p)$ as also the method of calculating the table. In Paper III, equation (C_1), we have

$$(p, q)=\Delta_1(p)\ \Delta_1(q)+\Delta_2(p)\ \Delta_2(q)+\dots+\Delta_p(p)\ \Delta_p(q)$$

where p is a positive integer not less than q. Thus from the table of of values of $\Delta_r(p)$ and equation (C_1) we can calculate $(o, p)_r$ and $(o, o)_r$.

2. We will use the abbreviations w and w_r for $\frac{1}{2}(o, o)$ and $\frac{1}{2}(o, o)_r$. and introduce an interesting co-efficient $\Delta_r(\omega)$, which is defined equal to

$$\begin{vmatrix} (1, 1), & (1, 2), & \cdot, & (1, r-1), & w_1 \\ (2, 1), & (2, 2), & \cdot, & (2, r-1), & w_2 \\ \cdot & \cdot & & \cdot & \cdot \\ (r, 1), & (r, 2), & \cdot, & (r, r-1), & w_r \end{vmatrix} \div [1, 2, \cdot, r]\ [1, 2, \cdot, r-1] \qquad (ii)$$

which may be more compactly written as

$$\frac{[1, 2, \cdot, r \mid 1, 2, \cdot, r-1, w]}{[1, 2, \cdot, r]\ [1, 2, \cdot, r-1]}.$$

Here it may be noticed that $w_1, w_2, \dots, w_r$ are not successive derivatives of w with respect to t but are derived from w in the manner already explained. In fact, we have by definition

$$w_r=r\ (1, r-1)+\frac{r(r-1)}{1.2}\ (2, r-2)+\dots+r\ (r-1, 1)$$

$$w_{r+1}=(r+1)\ (1, r)+\frac{(r+1)r}{1.2}\ (1, r-1)+\dots+(r+1)\ (r, 1)$$

Therefore

$$w'_r=r\ (1, r)+\frac{r\ (r-1)}{1.2}(2, r-1)+\dots+r\ (r-1, 2)$$

$$+r\ (2, r-1)+\dots+\frac{r\ (r-1)}{1.2}\ (r-1, 2)+r\ (r, 1)$$

$$=r\ (1, r)+\frac{(r+1)r}{1.2}\ (2, r-1)+\dots+\frac{(r+1)r}{1.2}\ (r-2, 2)+r(r,1)$$

$$=w_{r+1}-(1, r)-(r, 1)=w_{r+1}-2\ (1, r). \qquad (iii)$$

Thus the first derived of w_r with respect to t is not w_{r+1} but $w_{r+1}-2\,(1, r)$.

If $w_2, w_3, \cdot, w_{r+1}$ were the first derived of $w_1, \cdot, w_r$, respectively, with respect to s then we could apply formula E of paper III, namely

$$\Delta'_r(\omega) = \Delta_{r+1}(\omega).\ u_r - \Delta_{r-1}(\omega).\ u_{r-1}$$

$$\text{or } \Delta_{r+1}(\omega) = \frac{\Delta'_r(\omega) + \Delta_{r-1}(\omega).\ u_{r-1}}{u_r} \qquad \text{(E)}$$

by which we could successively and rapidly calculate $\Delta_3(\omega)$, $\Delta_4(\omega)$, $\Delta_5(\omega), \ldots,$ knowing $\Delta_1(\omega)$ and $\Delta_2(\omega)$.

Fortunately, however, we can shew that Formula E holds even when $w_2, w_3, \cdot, w_r$, are not the successive derivatives of $w_1, w_2, \cdot, w_r$, respectively, but are defined in the way we have already done.

We have

$\Delta_r(\omega) = c_1 w_1 + c_2 w_2 + \ldots + c_r w_r$, by (*ii*)

where $c_1, c_2, \cdot, c_r$ do not involve w.

Therefore

$$\begin{aligned} \Delta'_r(\omega) &= c'_1 w_1 + c'_2 w_2 + \ldots + c'_r w_r \\ &\quad + c_1 w'_1 + c_2 w'_2 + \ldots\ldots c_r w'_r \\ &= c'_1 w_1 + c'_2 w_2 + \ldots + c'_r w_r \\ &\quad + c_1 w_2 + c_2 w_3 + \ldots + c_r w_{r+1} \\ &\quad - 2\, c_1 (1,1) - 2\, c_2 (1,2) - \ldots - 2\, c_r (1,r) \quad \text{by } (iii) \\ &= c'_1 w_1 + c'_2 w_2 + \ldots + c'_r w_r \\ &\quad + c_1 w_2 + c_2 w_3 + \ldots + c_r w_{r+1} \end{aligned}$$

For, $c_1(1,1) + c_2(1,2) + \ldots + c_r(1,r)$

$$= \begin{vmatrix} (1,1), & (1,2), & \cdot, & (1,r-1), & (1,1) \\ (2,1), & (2,2), & \cdot, & (2,r-1), & (2,1) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ (r,1), & (r,2), & \cdot, & (r,r-1), & (r,1) \end{vmatrix} \div [1,2,\cdot,r]\,[1,2,\cdot,r-1] \text{ by } (ii)$$

$= 0$

Thus $\Delta'_r(\omega)$ would be obtained from $\Delta_r(\omega)$, if we simply wrote $w_2, w_3, \cdot, w_{r+1}$ in the place of the first derived of $w_1, w_2, \cdot, w_r$, with respect to s, in the process of differentiation. Formula E therefore holds for $\Delta_r(\omega)$ and enables us to calculate $\Delta_3(\omega)$, $\Delta_4(\omega)$, $\Delta_5(\omega), \ldots$, from knowledge of $\Delta_1(\omega)$ and $\Delta_2(\omega)$.

It may be observed that by Formula B_1 of Paper III we have

$$w_r = \Delta_1(r)\ \Delta_1(\omega) + \Delta_2(r)\ \Delta_2(\omega) + \ldots + \Delta_r(r)\ \Delta_r(\omega).$$

Thus w_r may be also calculated from tables of values of $\Delta_r(p)$ and $\Delta_r(\omega)$. A table of values of $\Delta_r(p)$ has been already given with Paper IV. The method of working out a table of values for $\Delta_r(\omega)$ we proceed to consider.

We will take the variable parameter t to be the arc length s, so that $[1]=1$, where $[1]=\sqrt{(1,1)}=\sqrt{\Sigma v_r'^2}$.

Now $\Delta\ (\omega)=\dfrac{w_1}{[1]}=o$ (*Vide* Paper III, Arts 1 & 2)

$$\text{and } \Delta_2(\omega) = \begin{vmatrix} (1,1), & w_1 \\ (1,2), & w_2 \end{vmatrix} \div [1]\ [1,2] = \frac{1}{u_1}$$

For, $w_1 = \frac{1}{2}\,(0,0)_1 = \frac{1}{2}.\ 0 = o$

$w_2 = \frac{1}{2}\,(0,0)_2 = (1,1) = 1,$

$(1,2) = \frac{1}{2}\,(1,1)' = o$

and u_1 (the first curvature) $=\dfrac{[1,2]}{[1]^3} = [1,2].$

$$\text{Therefore, } \Delta_3(\omega) = \frac{\Delta'_2(\omega) + \Delta_1(\omega).\ u_1}{u_2}$$

$$= -\frac{u'_1}{u_1^2 u_2}$$

$$\Delta_4(\omega) = \frac{\Delta'_3(\omega) + \Delta_2(\omega).\ u_2}{u_3}$$

$$= \frac{-u_1 u''_1\ u_2 + 2u'^2_1 u_2 + u_1\ u'_1\ u'_2 + u^2_1\ u^3_2}{u^3_1\ u^2_2\ u_3}$$

and so on.

3. An equation of the form

$$\Sigma \left\{ (X_s - x_s)^2 + 2\,A_s\,(X_s - x_s) \right\} = 0,\ (s = 1, 2, ., n) \qquad (iv)$$

represents what we may call a general n—*spheric* through the point x. The n co-efficients will become determined if the n—spheric is made to pass through any additional n points which are independent. Thus an n—spheric is uniquely determined by $n+1$ independent points. A system of such n—spherics will pass through $r+1$ points where r is less than n. The common intersection of the system of n—spherics will be an r—*spheric* through the $r+1$ given points, having its centre in the r—space determined by the $r+1$ given points.

If x be a point on a curve in an n—space, the osculating n—spheric at x will pass through x and an additional n points on the curve consecutive to x. Similarly the osculating r—spheric at x will pass through x and an additional r points ($r \angle n$) consecutive to x. The co-efficients A_s are determined for the osculating n—spheric by differentiating (iv) with respect to t, n times successively, considering X variable and x constant, and *then* putting $X = x$ in the result of each differentiation. We thus obtain

$$\left.\begin{array}{l} w_1 + \Sigma A_s\, x'_s = 0 \\ w_2 + \Sigma A_s\, x''_s = 0 \\ \dots\dots\dots\dots\dots\dots\dots \\ w_n + \Sigma A_s\, x^{(n)}_s = 0 \end{array}\right\} (v)$$

The result of eliminating the n co-efficients A_s between (iv) and (v) is

$$\begin{vmatrix} \Sigma\,(X_s - x_s)^2 & , 2\,(X_1 - x_1) & , 2\,(X_2 - x_2) & , & . & . & , 2\,(X_n - x_n) \\ w_1 & , \quad x'_1 & , \quad x'_2 & , & . & . & , \quad x'_n \\ w^2 & , \quad x''_1 & , \quad x''_2 & , & . & . & , \quad x''_n \\ . & , \quad . & , \quad . & , & . & , & \quad . \\ w_n & , \quad x^{(n)}_1 & , \quad x^{(n)}_2 & , & . & . & , \quad x^{(n)}_n \end{vmatrix} = 0 \qquad (vi)$$

which is the equation of the osculating n—spheric to the given curve at x.

If equation (vi) be referred to the n principal axes at x it takes the form

$$\begin{vmatrix} \Sigma\xi_s^2 & , & 2\xi_1 & , & 2\xi_2 & , & \cdot & , & 2\xi_n \\ w_1 & , & \Delta_1(1) & , & \Delta_2(2) & , & \cdot & , & \Delta_n(1) \\ w_2 & , & \Delta_1(2) & , & \Delta_2(2) & , & \cdot & , & \Delta_n(2) \\ \cdot & , & \cdot & , & \cdot & , & \cdot & , & \cdot \\ w_n & , & \Delta_1(n) & , & \Delta_2(n) & , & \cdot & , & \Delta_n(n) \end{vmatrix} = 0 \qquad (vii)$$

a result which may be obtained by multiplying (vi) by

$$\begin{vmatrix} 1 & , & 0 & , & 0 & , & \cdot & , & 0 \\ 0 & , & \Delta_1(x_1) & , & \Delta_1(x_2) & , & \cdot & , & \Delta_1(x_n) \\ 0 & , & \Delta_2(x_1) & , & \Delta_2(x_2) & , & \cdot & , & \Delta_2(x_n) \\ \cdot & , & \cdot & , & \cdot & , & \cdot & , & \cdot \\ 0 & , & \Delta_n(x_1) & , & \Delta_n(x_2) & , & \cdot & , & \Delta_n(x_n) \end{vmatrix}$$

vide—equation $\propto$ Paper III.

A Δ—transformation of the last n rows of (vii) gives

$$\begin{vmatrix} \Sigma\xi_s^2 & , & 2\xi_1 & , & 2\xi_2 & , & \cdot & , & 2\xi_n \\ \Delta_1(\omega) & , & 1 & , & 0 & , & \cdot & , & 0 \\ \Delta_2(\omega) & , & 0 & , & 1 & . & \cdot & , & 0 \\ \cdot & , & \cdot & , & \cdot & , & \cdot & , & \cdot \\ \Delta_n(\omega) & , & 0 & , & 0 & , & \cdot & , & 1 \end{vmatrix} = 0$$

$$\text{or } \Sigma\xi_s^2 - 2\xi_1\,\Delta_1(\omega) - 2\xi_2\,\Delta_2(\omega) - \ldots - 2\xi_n\,\Delta_n(\omega) = 0 \qquad (viii)$$

Hence the co-ordinates of the centre of the osculating n—spheric, referred to the n principal axes at x, are

$$\Delta_1(\omega),\ \Delta_2(\omega),\ \ldots,\ \Delta_n(\omega)$$

and the radius R_n is given by

$$R_n^2 = \Delta_1^2(\omega) + \Delta_2^2(\omega) + \ldots + \Delta_n^2(\omega), \qquad (i\ )$$

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a solution which in point of completeness and elegance leaves apparently nothing to be desired. We have already shewn how to express with facility $\Delta_1(\omega)$, $\Delta_2(\omega)$, &c in terms of the principal curvatures u_1, u_2, &c and their derivatives.

4. The equations of the *osculating r—spheric* are obtained in the same way. They are

$$\left\| \begin{matrix} \Sigma_1^n (X_s - x_s)^2 \, , & 2(X_1 - x_1) \, , & 2(X_2 - x_2) \, , & \cdot \, \cdot \, , & 2(X_n - x_n) \\ w_1 \, , & x'_1 \, , & x'_2 \, , & \cdot \, \cdot \, , & x'_n \\ w_2 \, , & x''_1 \, , & x''_2 \, , & \cdot \, \cdot \, , & x''_n \\ \cdot \, , & \cdot \, , & \cdot \, , & \cdot \, \cdot \, , & \cdot \\ w_r \, , & x_1^{(r)} \, , & x_2^{(r)} \, , & \cdot \, \cdot \, , & x_n^{(r)} \end{matrix} \right\| = 0 \quad (x)$$

where r is less than n.

These equations include the equations of the osculating r—space at x,

$$\left\| \begin{matrix} X_1 - x_1 \, , & X_2 - x_2 \, , & \cdot \, , & X_n - x_n \\ x'_1 \, , & x'_2 \, , & \cdot \, , & x'_n \\ x''_1 \, , & x''_2 \, , & \cdot \, \cdot \, , & x''_n \\ \cdot \, , & \cdot \, , & \cdot \, , & \cdot \\ x_1^{(r)} \, , & x_2^{(r)} \, , & \cdot \, , & x_n^{(r)} \end{matrix} \right\| = 0 \quad (xi)$$

of which the equations referred to the principal axes at x are

$$\xi_{r+1} = 0 \, , \quad \xi_{r+2} = 0 \, , \quad \cdot \, , \quad \xi_n = 0 \, ,$$

for, the osculating r—space is the join of the first r principal axes at x.

If we multiply (x) by

$$\left\| \begin{matrix} 1 \, , & 0 \, , & 0 \, , & \cdot \, , & 0 \\ 0 \, , & \Delta_1(x_1) \, , & \Delta_1(x_2) \, , & \cdot \, \cdot \, , & \Delta_1(x_n) \\ 0 \, , & \Delta_2(x_1) \, , & \Delta_2(x_2) \, , & \cdot \, \cdot \, , & \Delta_2(x_n) \\ \cdot \, , & \cdot \, , & \cdot \, , & \cdot \, , & \cdot \\ 0 \, , & \Delta_r(x_1) \, , & \Delta_r(x_2) \, , & \cdot \, \cdot \, , & \Delta_r(x_n) \end{matrix} \right\| \quad (xii)$$

that is if we multiply each determinant in (x) by the corresponding determinant in (xii) and take the sum of all the products, we have

$$\begin{vmatrix} \sum_1^r \xi_s^2 & , & 2\xi_1 & , & 2\xi_2 & , & \cdot & , & 2\xi_r \\ w_1 & , & \Delta_1(1) & , & \Delta_1(2) & , & \cdot & , & \Delta_2(r) \\ w_2 & , & \Delta_2(1) & , & \Delta_2(2) & , & \cdot & , & \Delta_2(r) \\ \cdot & , & \cdot & , & \cdot & , & \cdot & , & \cdot \\ w_r & , & \Delta_r(1) & , & \Delta_r(2) & , & \cdot & , & \Delta_r(r) \end{vmatrix} = 0 \quad (xiii)$$

where $\sum\limits_1^r \xi_s^2$ has been written instead of $\sum\limits_1^n \xi_s^2$, as $\xi_s = 0$ for $s = r+1 \;,\; \cdot \;,\; n$.

From ($xiii$) we obtain, by Δ—transformation of the last r rows,

$$\begin{vmatrix} \sum\limits_1^r \xi_s^2 & , & 2\xi_1 & , & 2\xi_2 & , & \cdot & , & 2\xi_r \\ \Delta_1(\omega) & , & 1 & , & 0 & , & \cdot & , & 0 \\ \Delta_2(\omega) & , & 0 & , & 1 & , & \cdot & , & 0 \\ \cdot & , & \cdot & , & \cdot & , & \cdot & , & \cdot \\ \Delta_r(\omega) & , & 0 & , & 0 & , & \cdot & , & 1 \end{vmatrix} = 0$$

or

$$\sum_1^r \xi_s^2 - 2 \sum_1^r \Delta_s(\omega) \; \xi_s = 0 \qquad (xiv)$$

which together with $\xi_{r+1} = 0 \;,\; \cdot \;,\; \xi = 0$

constitute the equations of the osculating r—spheric at x.

We have therefore the co-ordinates of the centre of the osculating r—spheric, referred to principal axes at x, given by

$$\Delta_1(\omega) \;,\; \Delta_2(\omega) \;,\; \cdot \;,\; \Delta_r(\omega) \;,\; 0 \;,\; \cdot \;,\; 0$$

and the Radius R_r given by

$$R_r^2 = \Delta_1^2(\omega) + \Delta_2^2(\omega) + \ldots + \Delta_r^2(\omega) \; .$$

We have evidently

$$R_r^2 - R_{r-1}^2 = \Delta_r^2 (\omega)$$

Also because $\Delta_r(\omega) \cdot u_{r-1} = \Delta'_{r-1}(\omega) + \Delta_{r-2}(\omega) \cdot u_{r-2}$

$$\text{or, } \Delta_r(\omega) \cdot \Delta_{r-1}(\omega) \cdot u_{r-1} - \Delta_{r-1}(\omega)\, \Delta_{r-2}(\omega) \cdot u_{r-2}$$
$$= \Delta_{r-1}(\omega) \cdot \Delta'_{r-1}(\omega)$$

We have, by putting successively r equal to 3, 4, . , r in the above equation and adding,

$$\Delta_r(\omega) \cdot \Delta_{r-1}(\omega) \cdot u_{r-1} - \Delta_2(\omega) \cdot \Delta_1(\omega) \cdot u_1$$
$$= \Delta_2(\omega) \cdot \Delta'_2(\omega) + \Delta_3(\omega) \cdot \Delta'_3(\omega) + \ldots + \Delta_{r-1}(\omega)\, \Delta'_{r-1}(\omega)$$

or, since $\Delta_1(\omega) = 0$, we have

$$\Delta_r(\omega) \cdot \Delta_{r-1}(\omega) \cdot u_{r-1} = \Delta_1(\omega)\, \Delta'_1(\omega) + \Delta(\omega)_2 \cdot \Delta'_2(\omega) + \cdot$$
$$+ \Delta_{r-1}(\omega)\, \Delta'_{r-1}(\omega).$$

$$\text{But } R_{r-1}^2 = \Delta_1^2(\omega) + \Delta_2^2(\omega) + \cdot + \Delta_{r-1}^2(\omega)$$

Therefore

$$R_{r-1}\, R'_{r-1} = \Delta_r(\omega)\, \Delta_{r-1}(\omega) \cdot u_{r-1} \qquad (xv)$$

which measures the rate of variation of R'_{r-1}.

On the vibrating string with an infinite number of Edges.

BY

GANESH PRASAD.

The object of this paper is to show how the transversal motion of a string can be investigated in the case when there are an *infinite* number of edges in the string. It is believed that this case has not been studied by any previous writer.

The method used is based on Harnack's treatment of the case of a finite § number of edges.

For the sake of convenience, I formulate the results as theorems. Also, Harnack's treatment of the case of a finite number of edges is reproduced for ready reference.

Harnack's Treatment of the Vibrating String with finite number of Edges.

1. Before I formulate and prove the theorem for the case of the vibrating string with a finite number of edges, it will be convenient to state the conditions * which obtain at an edge. It is obvious that here the partial differential equation is meaningless, for $\frac{\delta^2 y}{\delta x^2}$ is non-existent.

2. Let c denote the abscissa of an edge E, and let y_1 or y_2 denote the ordinate of a point according as it is to the left or to the right of E. Then, in a small period of time τ, an element $\frac{dc}{dt}\tau$ about the edge E undergoes a change in its momentum equal to

$$m \frac{dc}{dt} \tau . \left\{ \frac{\delta y_1}{\delta t} - \frac{\delta y_2}{\delta t} \right\},$$

m being the linear density of the string and $\frac{dc}{dt}$ being postulated to be existent and finite.

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§ A. Harnack, "*Ueber die mit Ecken behafteten Schwingungen gespannter Saiten*" (Math. Ann. Bd. 29).

* E. B. Christoffel, "*Untersuchungen über die mit dem Fortbestehen linearer partieller Differentialgleichungen verträglichen Unstetigkeiten*" (Ann. di Mat. s, 2, t. 8, 1876).

This change is brought about by the impulse

$$T\left\{\frac{\delta y_2}{\delta x}-\frac{\delta y_1}{\delta x}\right\}$$

acting for the period of time τ. Therefore,

since $\frac{T}{m}=1$, the units being suitably chosen, at E,

$$\frac{dc}{dt}\left\{\frac{\delta y_1}{\delta t}-\frac{\delta y_2}{\delta t}\right\}=\left\{\frac{\delta y_2}{\delta x}-\frac{\delta y_1}{\delta x}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(a)$$

3. Another equation is obtained from the consideration that, the displacement y being a continuous function,

$$y_1(c,t)=y_2(c,t).$$

which equation gives on differentiation,

$$\frac{\delta y_1}{\delta c}\frac{dc}{dt}+\frac{\delta y_1}{\delta t}=\frac{\delta y_2}{\delta c}\frac{dc}{dt}+\frac{\delta y_2}{\delta t},$$

$$i.\,e.,\ \frac{\delta y_1}{\delta t}-\frac{\delta y_2}{\delta t}=\frac{dc}{dt}\left\{\frac{\delta y_2}{\delta x}-\frac{\delta y_1}{\delta x}\right\}\qquad(b)$$

Combining (a) and (b) we find that $\frac{dc}{dt}=1$ and that, at the edge E,

$$\frac{\delta y_1}{\delta t}-\frac{\delta y_2}{\delta t}=\frac{\delta y_2}{\delta x}-\frac{\delta y_1}{\delta x},$$

$$i.\,e.\quad\left(\frac{\delta y}{\delta t}\right)_{c-0}-\left(\frac{\delta y}{\delta t}\right)_{c+0}=\left(\frac{\delta y}{\delta x}\right)_{c+0}-\left(\frac{\delta y}{\delta x}\right)_{c-0}\qquad(A)$$

4. *Theorem I.: If $y(x,t)$ is throughout continuous and, at every point other than an edge, the partial differential coefficients of $y(x,t)$ of the first two orders are finite and continuous, then the most general motion of the string is given by*

$$y(x,t)=\sum_{1}^{\infty}(A_n\cos nt+B_n\sin nt)\sin nx.$$

Proof:—

Since $\frac{\delta y}{\delta x}$ is existent and finite everywhere, with the exception of a finite number of points where it is non-existent, $y(x,t)$ must be expansible in the form

$$y(x,t)=\sum_{1}^{\infty}a_n\,(t)\sin nx,$$

where

$$a_n(t) = \frac{2}{\pi}\int_0^{\pi} y(x,t)\sin nx\,dx.$$

Now $\int^{\pi}\frac{\delta^2 y}{\delta x^2}\sin nx\,dx$ exists and is equal to

$$\int_0^{c_1} + \int_{c_1}^{c_2} + \dots + \int_{c_i}^{c_{i+1}} + \dots + \int_{c_{m-1}}^{c_m} + \int_{c_m}^{\pi}\frac{\delta^2 y}{\delta x^2}\sin nx\,dx,$$

where $c_1, c_2 \dots c_m$ are the abscissæ of the m edges at time t. Integrating by parts, we have

$$\int_{c_i}^{c_{i+1}}\frac{\delta^2 y}{\delta x^2}\sin nx\,dx = \left\{\sin n\,c_{i+1}\left(\frac{\delta y}{\delta x}\right)_{c_{i+1}-0} - \sin nc_i\left(\frac{\delta y}{\delta x}\right)_{c_i+0}\right\}$$

$$-n\left\{y_{c_{i+1}}\cos n\,c_{i+1} - y_{c_i}\cos n\,c_i\right\} - n^2\int_{c_i}^{c_{i+1}} y\sin nx\,dx.$$

Therefore, remembering that y is throughout continuous, we have

$$\int_0^{\pi}\frac{\delta^2 y}{\delta x^2}\sin nx\,dx = \sum_{i=1}^{i=m}\sin nc_i\left\{\left(\frac{\delta y}{\delta x}\right)_{c_i-0} - \left(\frac{\delta y}{\delta x}\right)_{c_i+0}\right\}$$

$$-n^2\int_0^{\pi} y\sin nx\,dx. \qquad (1)$$

Again

$$\frac{d}{dt}\int_0^{\pi} y\sin nx\,dx = \frac{d}{dt}\int_0^{c_1} y\sin nx\,dx + \sum_{i=1}^{i=m-1}\frac{d}{dt}\int_{c_i}^{c_{i+1}} y\sin nx\,dx$$

$$+\frac{d}{dt}\int_{c_m}^{\pi} y\sin nx\,dx.$$

But, since $\frac{dc_{i+1}}{dt} = \frac{dc_i}{dt} = 1$,

$$\frac{d}{dt}\int_{c_i}^{c_{i+1}} y \sin nx \, dx \, dx = y(c_{i+1}, t) \sin nc_{i+1} - y(c_i, t) \sin nc_i$$

$$+\int_{c_i}^{c_{i+1}} \frac{\delta y}{\delta t} \sin nx \, dx.$$

Hence, because of the continuity of y,

$$\frac{d}{dt}\int_0^\pi y \sin nx \, dx = \int_0^\pi \frac{\delta y}{\delta t} \sin nx \, dx.$$

In a similar manner, we get

$$\frac{d^2}{dt^2}\int_0^\pi y \sin nx \, dx = \sum_{i=1}^{i=m} \left\{ \left(\frac{\delta y}{\delta t}\right)_{c_i-0} - \left(\frac{\delta y}{\delta t}\right)_{c_i+0} \right\} \sin nc_i$$

$$+\int_0^\pi \frac{\delta^2 y}{\delta t^2} \sin nx \, dx. \tag{2}$$

Now, at every point other than an edge,

$$\frac{\delta^2 y}{\delta t^2} = \frac{\delta^2 y}{\delta x^2}.$$

Therefore

$$\int_0^\pi \frac{\delta^2 y}{\delta t^2} \sin nx \, dx = \int_0^\pi \frac{\delta^2 y}{\delta x^2} \sin nx \, dx,$$

Hence, using (1), (2),

$$\frac{d^2}{dt^2}\int_0^\pi y \sin nx \, dx = -n^2 \int_0^\pi y \sin nx \, dx + \sum_{i=1}^{i=m} \sin nc_i$$

$$\left\{ \left(\frac{\delta y}{\delta x}\right)_{c_i-0} - \left(\frac{\delta y}{\delta x}\right)_{c_i+0} \right\}$$

$$+\sum_{i=1}^{i=m}\left\{\left(\frac{\delta y}{\delta t}\right)_{c-o}-\left(\frac{\delta y}{\delta t}\right)_{c+o}\right\}\sin nc,$$

$$=-n^2\int_{0}^{\pi} y\sin n\mathrm{x}\,d\mathrm{x}$$

because of (A) of Art 3.

Therefore

$$\frac{d^2a_n}{dt^2}=-n^2a_n\,.$$

Therefore the most general value of a_n is $\mathrm{A}_n\cos nt+\mathrm{B}_n\sin nt$.

Q. E. D.

Treatment of the Vibrating String with infinite number of Edges.

5. *Theorem II.: If* y (x,t) *is throughout continuous and, at every point other than an edge, the partial differential coefficients of* $y(\mathrm{x},t)$ *of the first two orders are finite and continuous, then the most general motion of the string is given by*

$$y(\mathrm{x},t)=\sum_{1}^{\infty}(\mathrm{A}_n\cos nt+\mathrm{B}_n\sin nt)\sin n\mathrm{x},$$ *provided that*

(1) *the edges divide the string into an enumerable set of abutting intervals (having the edges as end-points), and*

$$(2)\quad\sum_{1}^{\infty}\left|\left(\frac{\delta y}{\delta \mathrm{x}}\right)_{c,-0}-\left(\frac{\delta y}{\delta \mathrm{x}}\right)_{c,+0}\right|$$

is convergent.

Proof.—Since $\frac{\delta y}{\delta \mathrm{x}}$ is existent and finite everywhere, with the exception of the edges at each of which the progressive and the regressive differential co-efficients are existent and finite, y (x,t) must* be expansible in the form

$$y\,(\mathrm{x},t)=\sum_{1}^{\infty}a_n\,(t)\sin n\mathrm{x},\ \text{where}$$

$$a_n(t)=\frac{2}{\pi}\int_{0}^{\pi}y\,(\mathrm{x},t)\sin n\mathrm{x}\,d\mathrm{x}.$$

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* Hobson, "*The theory of functions of a real variable and the theory of Fourier's series*," p. 681.

Now, since the edges form a set which is enumerable and is, consequently,[**] of measure zero, $\int_0^\pi \frac{\delta^2 y}{\delta x^2} \sin nx\, dx$ and $\int_0^\pi \frac{\delta^2 y}{\delta t^2} \sin nx\, dx$ exist ;[§] also, because of the proviso (2) of the theorem, the procedure, used in finding (1) and (2) of Art. 4, is applicable to the present case. Therefore

$$\int_0^\pi \frac{\delta^2 y}{\delta x^2} \sin nx\, dx = \sum_{i=1}^{i=\infty} \sin nc_i \left\{ \left(\frac{\delta y}{\delta x}\right)_{c_i-0} - \left(\frac{\delta y}{\delta x}\right)_{c_i+0} \right\}$$

$$-n^2 \int_0^\pi y \sin nx\, dx, \qquad (1)$$

and

$$\frac{d^2}{dt^2} \int_0^\pi y \sin nx\, dx = \sum_{i=1}^{i=\infty} \sin nc_i \left\{ \left(\frac{\delta y}{\delta t}\right)_{c_i-0} - \left(\frac{\delta y}{\delta t}\right)_{c_i+0} \right\}$$

$$+\int_0^\pi \frac{\delta^2 y}{\delta t^2} \sin nx\, dx. \qquad (2)$$

Now, at every point other than an edge, $\frac{\delta^2 y}{\delta t^2} = \frac{\delta^2 y}{\delta x^2}$.

Therefore, since the edges form a set of measure zero,

$$\int_0^\pi \frac{\delta^2 y}{\delta t^2} \sin nx\, dx = \int_0^\pi \frac{\delta^2 y}{\delta x^2} \sin nx\, dx.$$

Hence, using (1) and (2), given above, and (A) of Art. 3, we have

$$\frac{d^2}{dt^2} \int_0^\pi y \sin nx\, dx = -n^2 \int_0^\pi y \sin nx\, dx,$$

$$i.\ e.\ \frac{d^2 a_n}{dt^2} = -n^2 a_n.$$

[**] Ibid, p. 104.

[§] Ibid, p. 343.

Therefore the most general value of a_n is $A_n \cos nt + B_n \sin nt$.

Q. E. D.

6. A specially simple case of Theorem II is that in which the string vibrates in rectilinear strips. In this case, at every point other than an edge, $\frac{\delta^2 y}{\delta x^2}$ and, therefore, $\frac{\delta^2 y}{\delta t^2}$ vanish. Therefore $\frac{\delta y}{\delta t}$ is independent of t. Therefore, because of (A),

$$\left(\frac{\delta y}{\delta x}\right)_{c_i - 0} - \left(\frac{\delta y}{\delta x}\right)_{c_i + 0}$$

is independent of t. Therefore
the proviso (2) of Theorem II is satisfied if

$$\sum_{i=1}^{i=\infty} \left| f'(c_i - o) - f'(c_i + o) \right|$$

is convergent, $y = f(x)$ being the initial state of the string.

7. *Illustrative example.*

Let

$$c_i = \frac{\pi}{2} - \frac{\pi}{2^{i+1}} \text{ initially,}$$

$$f'(c_i - o) = \frac{1}{2^{i-1}},$$

$$\text{and } f'(c_i + o) = \frac{1}{2^i}.$$

Then, integrating by parts,

$$\int_0^\pi f(x) \sin n x \, dx = \frac{1}{n^2} \sum_{i=1}^{i=\infty} \left\{ f'(c_i - o) - f'(c_i + o) \right\} \sin nc_i$$

$$= \frac{1}{n^2} \sum_{i=1}^{i=\infty} \frac{\sin nc_i}{2^i}$$

$$= \frac{(-1)^{\frac{n}{2}-1}}{n^2} \sum_1^\infty \frac{\sin \frac{n\pi}{2^{i+1}}}{2^i} \text{ (for even } n\text{)}$$

$$\text{or } \frac{(-1)^{\frac{n-1}{2}}}{n^2} \sum_1^\infty \frac{\cos \frac{n\pi}{2^{i+1}}}{2^i} \text{ (for odd } n\text{).}$$

Thus

$$y\,(x,\,t)=\frac{2}{\pi}\sum_{n=1}^{n=\infty}\frac{C_n}{n^2}\cos nt \sin nx$$

gives the motion of the string, vibrating with the infinite number of edges, which initially formed the set $\left\{\frac{\pi}{2}-\frac{\pi}{2^{i+1}}\right\}$; C_n being equal to

$$(-1)^{\frac{n}{2}-1}\sum_{1}^{\infty}\frac{\sin\frac{n\pi}{2^{i+1}}}{2^{i}}\quad\text{or}\quad(-1)^{\frac{n-1}{2}}\sum_{1}^{\infty}\frac{\cos\frac{n\pi}{2^{i+1}}}{2^{i}},$$

according as n is even or odd.

FOURIER

His Life and Work.

By

A. C. Bose.

A very notable and informing Essay* written by a Mathematician last year introduces the founding of the Theory of Transfinite numbers in the following remarkable passage:—

"If it is safe to trace back to any *single man* the origin of those conceptions with which Pure Mathematical analysis has been chiefly occupied during the nineteenth century and *up to the present time,* we must trace it back to Jean Baptiste Joseph Fourier (1768—1830). Fourier was first and foremost a physicist and he expressed very definitely his view that mathematics only justifies itself by the help it gives towards the solution of physical problems and yet the light that was thrown on the general conceptions of a function and its "continuity" and of the "convergence" of infinite series, and of an "integral," first began to shine as a result of Fourier's original and bold treatment of the problems of conduction of heat. This it was that gave the impetus to the formation and development of the theories of functions. The broad-minded physicist will approve of this refining development of the mathematical methods which arise from physical conceptions, when he reflects that mathematics is a wonderfully powerful and *economically* contrived means, of dealing logically and conveniently with an immense complex of data, and that we can not be sure of logical soundness of our methods and results until we make every thing about them quite definite. The pure mathematician knows that pure mathematics has an end in itself which is more allied with philosophy. But we have not to justify pure mathematics here: we have only to point out its origin in physical conceptions. But we have also pointed out that physics can justify even the most modern developments of pure mathematics."

This very high appreciation of Fourier by a modern mathematician, whose masterly and critical writings on the history of the progress of mathematical thought, especially in the domain of the Theory of functions of a real variable, are so richly deserving of careful study, cannot fail to arouse our curiosity in regard to the Life and Labours of this eminent man. Alexander Freeman who nearly forty years ago, rendered a signal service to English readers by translating the monumental work of Fourier on the Analytical Theory of Heat, into English, deplored his inability, brought about by unforeseen circumstances, to prefix to his translation a Memoir of Fourier's life with some account of his other writings. This was indeed unfortunate and although the world values the possession of the two volumes, in French, of Fourier's collected works compiled by the great mathematician Darboux, the available materials, in the English language, for a suitable memoir are meagre. The Encyclopædia Britannica, Ninth Edition, which is enriched by a splendid contribution from the pen of Prof. Hobson on Fourier Analysis, devotes only about a column to the life of this original genius and the éloge of Arago, translated into English in 1857, like most obituary notices is rather "a panegyric than an impartial biography." Fourier's life has, therefore, yet to be written and what follows is but a feeble attempt to delineate a most remarkable career.

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* The Theory of Transfinite numbers—Cantor; with an Introduction and Notes by P. E. B. Jourdain, M.A. The open Court Publishing Co., 1915.

Birth and education of Fourier.

Fourier was born at Auxerre in Burgundy on the 21st March, 1768, the son of a tailor (Lambert, the German mathematician also, was the son of a tailor) and lost his parents at the early age of eight. By his precocious natural abilities and pleasant manners he attracted the notice of a lady who introduced him to the Bishop of Auxerre. Through his influence Fourier was admitted into the Military School at Auxerre which was, at that time, conducted by the Benedictines of the Convent of St. Mark. Like some other great mathematicians, Fourier developed early a taste for literature and the humanities and the progress made by him in such studies was surprising. It is said that "many a sermon very much applauded at Paris in the mouth of high dignitaries of the Church, were emanations from the pen of this school-boy of twelve." What truth there may be in this assertion it is difficult to say but it is certain that any one reading only the Preface to his Theory of Heat will find that he was an elegant writer—a master of good style. With his initiation, however, into the first principles of mathematics the character of the boy changed all at once. He felt that he had found his real vocation. "The hours prescribed for study no longer sufficed to gratify his insatiable curiosity. Ends of candles carefully collected in the kitchen, the corridors and the refectory of the college and placed on a hearth concealed by a screen, served during the night to illuminate the solitary studies by which Fourier prepared himself for those labours which were destined a few years afterwards, to adorn his name and his country." The writings of the Bernoullis, Euler, D'Alembert and Lagrange must have been studied by young Fourier and he must have drawn many an inspiration from the path-making researches of these great men whom Whewell has since characterised as the brilliant "Constellation of mathematicians."

Like Descartes, Fourier wished to be a soldier. His application, however, to sit for the examination for the artillery, though strongly supported by Legendre, was rejected. "Fourier" replied the Minister in charge, "not being *noble* could not enter the artillery, although he were a second Newton." Notwithstanding the odious character of the Minister's decision, events were soon to happen which enabled Fourier to renounce the profession of the Church which he was about to enter. But this circumstance did not prevent the authorities in the Military School at Auxerre from appointing him to the principal chair of mathematics. Fourier responded in a lively manner to this confidence and such was his culture and erudition that it is said that "when his colleagues were indisposed, he occupied in turn the chairs of rhetoric, of history and of philosophy diffusing among his audience the treasures of a profound mind adorned with all the brilliancy which the most elegant diction could impart to them."

The French Revolution—How it affected Fourier.

Let us glance for a moment at that great event in the history of France—the French Revolution.

Rousseau's *Social contract,* published in 1762, the theoretic basis of the Revolution, had filled men's minds with "beautifully seductive ideas relating to the social regeneration of France." The teachings of Voltaire had established the paramountcy of reason. The Commons of the States-General called themselves the National Assembly and claimed the Sovereignty as representing the People with whom the original contract was made. In June 1789 they at once adopted the basis of the British constitution, by refusing all taxation without their consent and with that refusal the Revolution began. In July 1789 the Bastille was stormed as the symbol of tyranny. The country rose against those whom they called "tyrants" and a Republic was proclaimed. The absurdity of privileged courts and classes supported by obsolete etiquette and mercenary troops, was exposed. There was a general diffusion of the sense of equal manhood and citizenship among all classes after the overthrow of the nobility's feudal privileges and the abolition of feudal serfdom. The close of the fateful year 1789 found young Fourier at Paris reading, in the midst of the clash of contending interests, before the Academy of Sciences, a memoir on the Resolution of numerical equations of all degrees, a subject on which he loved to dwell to the end of his days. On returning to Auxerre, he found "the town, the surrounding country and even the school to which he belonged, occupied intensely with the great questions relative to the dignity of human nature, philosophy and politics which were then discussed by the orators of the different parties of the National Assembly." "Fourier abandoned himself also to this movement of the human mind. He embraced with enthusiasm the principles of the Revolution and he ardently associated himself with every thing grand, just and generous which the popular impulse offered. His patriotism made him accept the most difficult missions." The Revolution had declared war against the enemies of the Republic and conscription had been introduced for the first time. As at the present day in the great and terrible drama which is being enacted on the continent of Europe, so at the close of the eighteenth century Men and Munitions were required. As a member of the popular society of Auxerre, Fourier was entrusted with the duty of recruitment. Fired by patriotism and enthusiasm, this young mathematician made the words "honour, country and God ring so eloquently" and so many enrolled themselves voluntarily that the ballot was not deemed necessary. The required levy of 3,000 men was obtained without any difficulty. "At the command of the orator the contingent assigned to the chief town of the Yonne formed in order and marched forthwith to the frontier."

How did science and Mathematics fare during the great cataclysm of the Revolution? To answer this question fully would be to rewrite the *Eloges Historiques* of Cuvier, the greatest representative of the Academic system—the never-to-be-forgotten permanent Secretary of the French Academy of Sciences. Suffice it to say that under the Revolution the work of Reconstruction proceeded side by side with the work of Demolition. Much of the good work done by the attempts of Fontenelle, Voltaire, Buffon, Condorcet and others to popularize science in France just before the Revolution, was "spoiled by premature and ill-founded theories," such as were embodied in La Mettrie's *Histoire Naturelle de l' Ame* and *L' Homme Machine* which were so much in favour with Frederick the Great or Holbach's "Systeme de la nature." But works like these were "brilliant but hasty generlizations" and the tide of materialism reached its highest level with Diderot, the Encyclopaedist—the friend of D' Alembert, the man of "fire and genius," who is so often called "the head and leader" of the Materialists. One effect of the Revolution was to demolish the influence of such teachings and to substitute in its place, in the words of John Theodore Merz, the Historian of European thought, "the modern practical popularization of Science and to establish its educational and its technical importance. Science was to be not an elegant amusement, or a refined luxury, nor even exclusively the serious occupation of the rare genius: it was to be the basis of *a national* instruction and the foundation of the greatness and wealth of the Nation. The Memoirs of the Academy were cleansed of all dangerous generalizations which might have brought them into touch with political controversy; the language was confined to the measured and concise statements of facts or to theories capable of mathematical verification and treatment; conjectural matter was carefully excluded and the standard of scientific excellence, both in matter and form, was raised, to which we still look up with admiration. At the same time this lofty and dignified spirit enlivened the courses of lectures delivered in the great Schools by the first men of the nation and became through them the habit of a large number of ardent pupils, who were to carry it further into more popular teaching or into the applications of art and industry. The results of both are well-known. If now we continually appeal to scientific authorities for aid in the solution of practical problems, it is well to remember that nothing helped more to raise science to the eminence of a great social power than the action of the Revolutionary Government in 1793. Whilst (on the one hand) it guillotined illustrious men like La Voisier, Bailly and Cousin; drove Condorcet to suicide and others into premature death, it had, on the other hand, to appeal for its most necessary requirements to the society of scientific authorities, which it professed not to need." Wonderful spectacle this! More wonderful was the display of patriotism on the part of the mathematicians and scientists of the time—which finds an exact parallel at the present day in the great theatre of the European war! "Coalesced Europe had launched against France a million of soldiers and it became necessary to organize for the crisis fourteen armies." Who was it who directed this gigantic operation? None but L. N. M. Carnot—the great author of *Essai Sur les Machines* and of the *Geometrie Des Positions*—whose profound discourse on the 'Metaphysics of the Infinitesimal Calculus, I have just before me as I write—a rare treat to the philosophically inclined! "Every thing was wanting for the defence of the coutnry—Powder, Cannon, Provisions. The Arsenals were empty, steel was no longer imported from abroad, salt-petre came not from India. It was exactly those men whose labours had been proscribed, who could give to France what she wanted." The savants rose to the occasion and responded. Chemists like Fourcroy and Berthollet, mathematicians like Carnot and Gaspard Monge, rushed to the defence of French independence; some of them extracted from the soil, by prodigies of industry, the very last atoms of salt-petre it contained some made known new methods of manufacturing gun-powder and studied the making of iron and steel; Monge, the incomparable mathematician, explained the art of casting and boring cannon of brass for land use and cast-iron cannon for the navy; at the voice of his country in danger, the young and learned Mennier, readily renounced the seductive pursuits of the laboratory, joined the army and was killed in action in the defence of Mayence, after attaining the highest position in the garrison to which he was attached."

Such were the times and such the sacrifices made by Frenchmen of all sorts and conditions! Fourier, a true child of the soil, could hardly remain idle. Although he did not actually gird on the sword and join the army of the Revolution, he made himself useful in other ways. While the Convention destroyed existing academies, colleges and universities because they supported "*Les Messieurs*," it soon resolved upon organising a system of public instruction. "But a difficulty arose in finding professors. The members of the Corps of Instruction had become officers of artillery, of engineering or of the staff and were combating the enemies of France at the Frontiers. Fortunately at this epoch of intellectual exaltation, nothing seemed impossible. Professors were wanting; it was resolved to *create* some and the Ecole Normale Superieure sprang into existence. Fifteen hundred citizens of all ages despatched from the principal district towns, assembled together, not to study the different branches of knowledge but to learn the art of teaching under the greatest masters. Fourier was one of these fifteen hundred pupils." Before Fourier became, thus, a pupil of the Ecole Normale, Arago tells us, he had passed through several adventures during the Reign of Terror. "He was often thrown into prison by the Decemvirs. He had offered before the revolutionary tribunal the assistance of his admirable talents, of sweet and persuasive eloquence (which led St. Just to style him a patriot in music), to the mother of Marshal Davoust accused of the crime of having at that unrelenting period sent some money to the emigrants. He had the incredible boldness to shut up at the Inn of Tonnerre an agent of the Committee of public safety and thus obtained time to warn an honourable citizen that he was about to be arrested. He had attached himself to the sanguinary proconsul before whom every one trembled in Yonne, made him pass for a mad man and obtained his recall. These were some of the acts of patriotism, of devotion and of humanity which signalized the early years of Fourier."

Lecturer at the Ecole Normale and the Ecole Centrale des Travaux Publics.

The year 1794 was signalized by the foundation of the two great national Schools, the Ecole Normale Superieure and the Ecole Centrale des Travaux Publics better known by the title Ecole Polytechnique. It has been said that "few of the higher and philanthropic aims of the great educational leaders of the early years of the Revolution, of Mirabeau, of Talleyrand, of Condorcet, were realizaed; little was done for primary education; but science can boast of having been worthily represented and supported in the two great schools which still bear their original description and which can show a record of celebrated names and magnificent work superior probably to that of any other institution in Europe." The founders recognised "that, in spite of the diversity of applications, mathematics and physics are the indispensable bases of the studies in view." The Ecole Normale lasted for four months and was closed under a decree of the Convention but was reopened in the year 1808 under the Empire by the same decree of 17th March which organized the University of France. Short as was the first period of the life of the Ecole Normale "we are indebted to it for the foundation of a new branch of science, *viz.*, the "Descriptive Geometry"—that fine creation of the genius of Monge—which was given to the world through short-hand notes from his Lectures delivered in that institution." It is interesting to note that the first professors in the mathematical, physical, and chemical sections were:—

Lagrange and Laplace ...	Mathematics.
Monge	Descriptive Geometry.
Haüy	Physics.
Berthollet	Chemistry.

With his great talents, from pupil to lecturer, was but a short step for Fourier. He was appointed a lecturer and "the precision, the clearness and the elegance of his lectures soon procured for him the unanimous applause of the fastidious and numerous audience" entrusted to his care.

As regards the mode of instruction at the Ecole Normale, Arago has given the following interesting description:—

"We see in the ancient great colleges, professors concealed in some degree behind their portfolios, reading as from a pulpit, amid the indifference and inattention of pupils, discourses, prepared beforehand with great labour, which reappear every year in the same form. Nothing of this kind existed at the Normal School: oral lessons alone were there permitted. The authorities even went so far as to require of the illustrious savans appointed to the task of instruction, the formal promise never to recite any lectures which they might have learnt by heart. From that time the Chair became a tribune where the professor, identified so to speak, with his audience, sees in their looks, in their gestures, in their countenance, some times the necessity of proceeding at greater speed, some times, on the contrary of retracing his steps, of awakening the attention by some incidental observations, of clothing in a new form, the thought which, when first expressed had left some doubts in the minds of his audience." The value of such a method of instruction cannot be sufficiently emphasised even at the present day!

His work at the École Normale established Fourier's reputation as a brilliant teacher and he was not forgotten by the founders of the Ecole Polytechnique, to which celebrated institution, he was at first attached with the title "Superintendent of Lectures on Fortification." Being afterwards appointed to deliver a course of lectures on analysis, Fourier left there "a venerated name" and the reputation of a lecturer distinguished by "clearness, method, erudition and grace." These lectures have not been collected together. The journal of the Polytechnic School contains only his memoir on the "Principle of virtual velocities." This memoir shows that the secret of Fourier's "great success consisted in the combination of abstract truth, of interesting applications and of historical details little known and derived from original sources." Those who have carefully read Fourier's published works will readily see that there is no exaggeration in this description.

Fourier's work in Egypt.

The story of how the Revolution ended and how Napoleon Bonaparte gradually rose into Power, this is neither the place nor the occasion to recapitulate. It is well-known that Napoleon "favoured the mathematical sciences and created great prizes for physical discoveries." This was so partly because he himself had had a mathematical training. During his campaigns in Italy and Germany and on his expedition to Egypt, he surrounded himself with some of the great scientists of the day *e.g.* Berthollet and Monge. Fourier, whose fame had spread far and wide, also accompanied Napoleon to Egypt. He sailed in the same ship with the celebrated Kléber and a fast friendship sprang up between these two men, the one, a philosopher and the other, a warrior. Napoleon lost no time in founding the Institute of Egypt of which Monge became the first President and Fourier the perpetual Secretary. His scientific work in Egypt was valuable. In the *Decade* and the *Courier* of Egypt we find "(i) a memoir upon the general solution of algebraic equations, (ii) researches on the methods of Elimination, (iii) the demonstration of a new theorem of Algebra, (iv) a memoir upon indeterminate analysis, (v) studies in general mechanics, (vi) a technical and historical work upon the aqueduct which conveys the waters of the Nile to Cairo, (vii) reflections upon the oases, (viii) a plan of statistical researches to be undertaken with respect to the State of Egypt, (ix) programme of an intended exploration of the site of the ancient Memphis and of the whole extent of burial places, (x) a descriptive account of the revolutions and manners of Egypt from very early times; and (xi) a description of a machine designed to promote irrigation and which was to be driven by the power of wind." These serve only to show the many-sidedness of Fourier's character and intellect. Besides his work as the perpetual Secretary of the Institute of Egypt, Fourier was entrusted with more difficult duties which brought him into contact with the local population and notabilities, as French Commissioner at the Divan of Cairo and in the performance of which he invariably displayed such suavity of manner and a scrupulous regard for prejudices, that he came to be held in great veneration by the Shiekhs and Ulemas. These friendly relations enabled Fourier to carry out successfully difficult political missions confided to him by the Generals and the Treaty with Mourad Bey was due to Fourier's tact and urbanity,—a treaty which involved also an element of romance, for, Mourad's plenipotentiary was a woman, Sitty Nafiçah, of "unparalleled beauty" whom General Kléber "has immortallized in his Bulletin of Heliopolis" and whose charms, it is said, had led to bloody revolutions among the Mamelukes. Again, the treaty with the fiery Janissaries who had seized upon Cairo while the war was raging at Heliopolis, was also concluded successfully by Fourier, not, as Arago puts it, "within the mysterious precincts of a harem, upon downy couches, under the shade of balmy groves, but in a house riddled with bullets and grape-shots" giving unmistakeable proof of Fourier's bravery and coolness in the midst of danger.

Fourier's return to France.

It is well-known how and why Napoleon's Egyptian Expedition failed; how on August 22, 1798 he wrote to General Kléber that he transferred to him the command of the Expedition and how he suddenly sailed for France taking only Monge and Berthollet among the scientists with him and leaving Fourier behind. Brave Kléber fell by the hand of an assasin and it was left to his intimate friend Fourier to deliver the funeral oration. And nobly it was delivered "upon the breach of a bastion" which the French troops had recently taken by assault "in sight of the most majestic of rivers, of the magnificent valley which it fertilizes, of the frightful desert of Lybia, of the colossal pyramids of Gizeh and in the presence of the most valiant soldiers that had ever set foot on a land wherein the names of Alexander and of Cæsar still resound." A few months later General Desaix sacrificed his life at Marengo to secure the triumph of the French armies and it was on the "same bastion and before the same soldiers, Fourier celebrated with his wonted eloquence, the exploits of the great General."

Fourier quitted Egypt with the wreck of the French Army after the capitulation signed by Menou. He had no reason to regret his life in Egypt, of nearly half of which he was virtually Governor. On his return to France he was made Prefect of Grenoble and was created Baron and Chevalier of the Legion of Honour. He wrote the Historical Preface to the *Description de l'Egypte* and Fontanes saw in that Preface "the graces of Athens and the wisdom of Egypt united together"—no small tribute to the literary merits of the work. He held the Prefecture for fourteen years in the course of which, the most memorable acts of public utility which he performed, were the construction of the great road from Grenoble to Turin by Mount Genevre and the drainage of the marshes of Bourgoin converting thereby "a focus of infection" into a land of "rich harvests, fat pastures, numerous flocks, a robust and happy population." Amidst all the exacting work of a difficult administration it is wonderful how Fourier found time, to carry on those elaborate and recondite investigations on the propagation of heat in solid bodies which at once established his fame as a physicist and mathematician of the first rank.

The Theory of heat.

It was at Grenoble also in the midst of his administrative duties as prefect of l'Isere, that Fourier composed his *Théorie Analytique de la Chaleur*—that monumental work of which M. Cousin said that "the grandeur of its results was no more to be questioned than their certainty" and "that in the opinion of scientific Europe the *novelty* of the analysis on which they rest is as evident as its completeness." This work was published in 1822 and "in its new methods and great results it made an epoch in the history of mathematical and physical science." Two years afterwards (1824), and also in France, another work was published on the Theory of Heat which has marked another epoch in the history of Physical Science. Sadi Carnot, the son of Lazare Carnot the mathematician of whom we have spoken above, published, in the form of a pamphlet, his *Reflexions Sur la puissance motrice du feu* which really enunciated the fundamental problem out of which arose the fruitful science of Thermo-dynamics and the great idea of Conservation of Energy—the idea which unified sciences hitherto far apart. Carnot was initially brought up in the School that looked upon heat as an imponderable substance "which might hide itself—might become latent but could not be created or destroyed" (N.B.—We know, however, from the publication of some of Carnot's posthumous papers by his brother H. Carnot in 1878, that Carnot, before he died, had abandoned the material theory of heat). Carnot's view as to the *nature* of heat was also the view of Black, Laplace and Fourier, although it was not the view of Cavendish, of Davy and of Rumford. But whatever might have been the *nature* of heat, whether it was a substance or whether it was a motion, the propositions enunciated by Carnot and Fourier, were independent of any particular hypothesis. As E. Mach has remarked "if we are astonished at the discovery that Heat is motion, we are astonished at some thing which has never been discovered. *It is quite irrelevant for scientific purposes whether we think of heat as a substance or not.*" Fourier's views are disclosed in his Preliminary Discourse to his great work. According to him, "the properties of heat formed a special order of phenomena, which were not to be explained by principles of motion and equilibrium;"—and that "there existed a very extensive class of phenomena which were not produced by mechanical forces but which resulted solely from the presence and accumulation of heat. This part of natural philosophy could not be brought under dynamical theories: it had principles peculiar to itself. The dilatations, indeed, caused by the repulsive force of heat, the observation of which dilatations, served as a measure of temperature, were dynamical effects but it was not these dilatations which we calculated when we investigated the laws of the propagation of heat." Fourier then built up, as he tells us, his new science "upon a very small number of simple facts, of which the *causes were unknown* but which were gathered by observation and confirmed by experiments." He arrived at certain relations, expressed by differential equations, different from, though analogous to the general equations of dynamics. One of the great experimental facts on which Fourier based his theory was that "all motion of heat depends on differences of temperature."

The other factors in the problem of the transference of heat were the following:—

(*i*) Power of bodies to *contain* heat.

(*ii*) Power of bodies to *receive or transmit* heat across their surfaces.

(*iii*) Power to *conduct* heat through the interior of their masses.

These were the three specific qualities which Fourier's theory clearly distinguished for the first time and showed how to measure.

Having obtained the differential equations of the propagation of heat under most general conditions and thereby having reduced the physical questions to problems of pure analysis (which was the proper object of the theory), Fourier speaks thus of the evolution of his Analysis:—

"The equations of the movement of heat, like those which express the vibrations of sonorous bodies, or the ultimate oscillations of liquids, belong to one of the recently discovered branches of analysis, which it is *very important to perfect.* After having established these differential equations, their integrals must be obtained; this process consists in passing from a common expression to a particular solution subject to all the given conditions. This difficult investigation requires a *special analysis founded on new theorems.*"

Fourier, before he gave to the world his mathematical theory, had, as he himself tells us, been engaged "in a prolonged study and attentive comparison of the facts known" up to his time and all these facts he had checked and verified in the course of several years with the most exact instruments then in use. He was thus, above all, as stated in the opening passage, a physicist. But he was no less a mathematician. And after telling us, as a physicist, that the "profound study of nature is the most fertile source of mathematical discoveries," he proceeds to speak of the scope and power of mathematical analysis as follows:—

"There cannot be a language more universal and more simple, more free from errors and from obscurities, that is to say, more worthy to express the invariable relations of natural things. Considered from this point of view, *mathematical analysis is as extensive as nature itself*; it defines all perceptible relations, measures times, spaces, forces, temperatures; this difficult science is formed slowly but it *preserves* every principle when it has once acquired it, it grows and strengthens itself incessantly in the midst of the many variations and errors of the human mind. Its chief attribute is clearness; it has no marks to express confused notions. It brings together phenomena the most diverse, and discovers the hidden analogies which unite them. If matter escapes us, as that of air and light, by its extreme tenuity, if bodies are placed far from us in the immensity of space, if man wishes to know the aspect of the heavens at successive epochs, separated by a great number of centuries, if the actions of gravity and of heat are exerted in the interior of the earth at depths which will be always inaccessible, mathematical analysis can yet lay hold of the laws of these phenomena. It makes them present and measurable and seems to be a faculty of the human mind, destined to supplement the shortness of life and the imperfection of the senses; and what is more remarkable it follows the same course in the study of all phenomena; it interprets them in the same language, as if to attest the *unity and simplicity of the plan of the universe* and to make still more evident that unchangeable order which presides over all natural causes."

As regards the problems of the Theory of heat which he solved by the beautiful analysis he discovered, Fourier says that "if the order which is established in these phenomena, could be grasped by our senses, it would produce in us an impression comparable to the sensation of musical sounds."

Fourier spoke as a mathematical physicist elated by his discovery but the verdict of posterity has upheld much of all that he has said. Maxwell called Fourier's Theory "a mathematical poem." Lord Kelvin drew his early inspirations from it. It was Professor Nichol who, in 1840, brought to the notice of his eager pupil, the Theory which was destined to influence his whole career. Lord Kelvin's own account is as follows:—

"The origin of my devotion to these problems is that after I had attended in 1839 Nichol's Senior Natural Philosophy class, I had become filled with the utmost admiration for the splendour and poetry of Fourier. Nichol was not a mathematician and did not profess to have really read Fourier but he was capable of perceiving his greatness. I asked Nichol if he thought I could read Fourier. He replied "perhaps." He thought the book a work of most transcendent merit. So on the 1st of May (1840), the very day when the prizes were given, I took Fourier out of the University Library; *and in a fortnight I had mastered it—gone right through it.*" So great was Lord Kelvin's (then William Thomson) enthusiasm and admiration for Fourier that he nearly quarrelled with Kelland, over a passage in the latter's *Theory of heat* (*Vide* Art. 70, page 64). Lord Kelvin gives an interesting account of this matter when he writes "going in summer to Germany with my father and my brothers and sisters I took Fourier with me. Now just two days before leaving Glasgow, I had got Kelland's book and was shocked to be told that Fourier was mostly wrong. So I put Fourier into my box and used in Frankfort to go down to the cellar surreptitionsly every day to read a bit of Fourier." The result was that the young analyst of sixteen, successfully vindicated Fourier's work from the charge of error brought by Kelland—a distinguished and experienced mathematician "who, in fact, had been misled by not comprehending that a Fourier series may be expressed either in a double series of sine and cosine terms, or in a single series of either sines or cosines, by *appropriate assignment of epochs to the various terms of the series.*" At Frankfort, Thomson discovered the cause of the misunderstanding and wrote his article "On Fourier's expansions of functions in Trigonometrical Series." As regards this paper Thomson says in a letter to Prof. Sylvanus Thompson—the future biographer of Lord Kelvin :—

"I was filled with indignation at a statement by Kelland that almost everything in Fourier was wrong. When I wrote my paper—my first published original paper—for the Cambridge Mathematical Journal, my father sent it to Gregory. Gregory had been beaten recently by Kelland in the competition for the Edinburgh chair of Mathematics. Gregory thought the paper rather controversial, and sent it to Kelland. This was a graceful act on Gregory's part, that he would not put it into the Journal without referring it first to Kelland. Kelland wrote back rather tartly, as if piqued. Then my father and I went over the paper and smoothed down a few passages that might have offended Kelland's feelings. Kelland wrote back that he was charmed with the paper, and was quite amiable. So then it was printed."

"All through his life Lord Kelvin continued to cherish his youthful enthusiasm for the men who had inspired him. Fourier and Green in the domain of mathematical physics and Faraday in that of experimental science, were the *Dimajores* of his veneration." And in his maturer years, his considered verdict as regards Fourier's *Theorie Analytique* and its solutions of heat problems was that" "it is difficult to say whether their uniquely original quality or their transcendent interest or their perenially important instructiveness for physical science, is most to be praised." If we turn to the "Mathematical and Physical Papers" of Lord Kelvin, we not only find that the very first few pages of the first volume contain the article in correction of Kelland, but also that the second volume contains a chapter entitled "A compendium of Fourier mathematics"—"admirable in itself but thrown together like mere hints to a lecturer"; we find too that the third volume contains a discussion of the equation which Fourier had found for the 'linear motion of heat,' which was applied by Lord Kelvin to other physical qualities viz:—(*i*) the velocity of motion as propagated in a viscous liquid, (*ii*) the diffusion of liquids into one another, (*iii*) the diffusion of induced electric currents into a homogeneous conductor, and (*iv*) the diffusion of electric potential in the conductor of a submarine cable; and thus the great generality and power of the methods of Fourier were demonstrated (C. F. Articles 243, 332, 333, 801-805 of Maxwell's *Electricity and Magnetism*.)

Fourier's Théorie was the finished production of the artist—the final result of his researches carried on previously for many years. We note, harking back for a moment to previous history, that as early as the year 1736, the Academy of Sciences of France, had proposed the solution of the Problem of heat as the subject of a prize essay. The terms "heat" and "caloric" were then not in use; the problem proposed was "the study of the nature and the propagation of *Fire*." The great Euler was one of the competitors and his thesis was crowned along with that of two others who were unknown to fame and were not mathematicians. Euler's main thesis was not distinguished by any brilliant analysis; but he appended a supplement to his memoir, a supplement "truly worthy of his genius." It is remarkable that even Voltaire, the author of Henriade and a lady of rank, the Marchioness du Chatelet, competed for the prize. Arago has an interesting note that the lady's memoir contained some valuable projects of experiments, one of which was since developed by Herschel and from which he derived "one of the principal flowers of his brilliant scientific crown." The mathematician Lambert next dealt with the longitudinal propagation of heat in a long bar heated at one end and discovered the correct relation between the final temperatures and the distances of the different particles of the bar from the extremity directly heated. But when investigators took up the general problem of the propagation of heat in a body the aid of the higher analysis was found to be indispensable. It was Fourier who first assigned the differential equation and it was Fourier also who gave certain theorems by which this equation could be solved and the solutions pushed, in the majority of cases, to final numerical calculations. Fourier's first memoir is dated 1807. It startled Lagrange by the novelty of its mathematical analysis which contained the "absolutely revolutionary doctrine" that an arbitrarily given curve or function irrespective of its nature could be represented in any interval by a trigonometric series. An extract was published in 1808 in the *Bulletin des sciences* of the Societe Philomatique. Notes were added to this Memoir by Fourier, from time to time, concerning the convergence of series, the diffusion of heat in an infinite prism, its emission in spaces void of air and the analysis of the periodic movement of heat at the surface of the earth. When the Academy of Sciences again made the Problem of the propagation of heat, the subject of a great mathematical prize for 1812, Fourier competed and submitted his second Memoir on the 28th of September 1811. This Memoir was crowned. Lagrange, Laplace, Legendre, Malus and Haüy who were the judges, gave their verdict as follows :—This work "renferme les véritables équations différentielles de la transmission "de la chaleur, soit á l' intérieur des corps, soit á leur surface ; et la "nouveauté du sujet, jointe á son importance, a détermine la Classe á "Couronner cet Ouvrage, en observant cependant que la maniére dont "l' Auteur parvient á ses équations n'est pas exempte de difficultés, "et que son analyse, pour les intégrer, laisse encore quelque chose á désirer, soit relativement á la généralité, soit méme du côté de la rigueur."

Alas! as Fontenelle said "In the country even of demonstrations, there are to be found causes of dissensions." While the illustrious Judges admitted the novelty and importance of the Memoir and while they acknowledged that th ereal differential equations had been finally found, they perceived difficulties "in the way in which the author arrived at them" and they added that "his process of integration left something to be desired, even on the score of *rigour*." The Judges, however, did not give their reasons for this qualified verdict.

Fourier, to the end of his life, never admitted the validity of this decision, which "poisoned the pleasure of his triumph." His highly susceptible nature could ill brook this criticism as to the rigour of his analysis. And he looked, with a certain degree of displeasure, on the subsequent labours of mathematicians to improve his theory. Arago views Fourier's attitude "as a strange aberration of a mind of so levated an order." He rightly says that Fourier forgot that "it was not allotted to any person to conduct a scientific question to a definite termination" and that "the important labours of D'Alembert, Clairaut, Euler, Lagrange and Laplace, while immortalizing their authors, have continually added new lustre to the imperishable glory of Newton." As in the case of Newton, so in the case of Fourier, the imperishable writings of a number of great mathematicians to improve Fourier's analysis on the side of rigour have not only justified in some measure, the verdict of the illustrious judges, but have served also to exhibit in a most interesting light the very *general* nature of Fourier's analysis, and to place it on a more solid foundation. The magician's wand, it is true, had raised a beautiful edifice on ground that concealed a marsh and the marsh had to be drained and replaced by reinforced concrete, so that the edifice might endure for all time—a lasting monument to the magician's power. And in thus making the foundations more secure, how transforming and useful have been the labours of the great leaders of mathematical thought!

On the motion of two spheroids in an infinite liquid along the common axis of revolution.

By

Bibhutibhushan Datta.

It is well-known that the problem of the motion of two spheres in an infinite liquid along the line joining their centres has been completely solved by various investigators. But the corresponding problem for two spheroids or ellipsoids has remained unsolved up to now; the only previous writer to attempt it with some measure of success being Prof. Karl Pearson.[*]

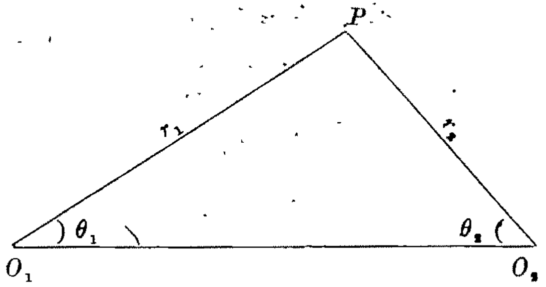
The object of this paper is to show how the problem can be solved in the case of two spheroids of revolution of *small* ellipticities, the motion of the solids being along their common axis of revolution. The method used is one of continued approximation.

In Art. 1, I investigate the approximate formal expressions for the velocity potential and the current function. In Art. 2, the first approximation to the velocity potential is obtained. Art. 3 gives two sets of algebraic equations from which, by continued approximations, we can determine the necessary constants and thus proceed to the complete solution. In Art. 4, I give the general values of the constants.

All the results in this paper are believed to be new.

I wish to express my indebtedness to Dr. Ganesh Prasad at whose suggestion I took up, and under whom I carried on, the investigation the results of which are embodied in this paper.

§ 1.



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[*] "On the motion of spherical and ellipsoidal bodies in fluid media," Part II, (*Quarterly Journal of Pure and Applied Mathematics*, Vol. XX).

Let (r_1, θ_1) and (r_2, θ_2) be the polar co-ordinates of a point P referred to two points O_1 and O_2; let s be the distance between them. Then we shall take $\frac{d}{ds_{12}}$ to denote differentiation in the direction of s measured from O_1 to O_2 and $\frac{d}{ds_{21}}$ to denote differentiation in the opposite direction, so that $ds_{12} = -ds_{21}$.

The problem before us is to find a function ϕ satisfying the conditions:—

$$(i) \qquad \nabla^2 \phi = o;$$

$$(ii) \qquad \phi = o, \text{ at infinity,}$$

$$(iii) \qquad \frac{d\phi}{dn_1} = -u_1 \cos\left(\overset{\wedge}{n_1\, s_{12}}\right) \text{ over the first solid,}$$

$$(iv) \qquad \frac{d\phi}{dn_2} = -u_2 \cos\left(\overset{\wedge}{n_2\, s_{21}}\right) \text{ over the second solid,}$$

where dn denotes an element of the normal to the surface drawn outward into the liquid, $\left(\overset{\wedge}{ns}\right)$ denotes the angle between the axis and the normal and u_1, u_2 are the velocities of the solids along the common axis of revolution.

Since the motion is symmetrical about the axis, we have another function ψ, called Stokes's current function,[*] and the boundary conditions (iii) and (iv) now take the forms,[†]

$$(iii^1) \qquad \psi = -\tfrac{1}{2} u_1 r_1^2 (1-\mu_1^2) + \text{constant,}$$

$$(iv^1) \qquad \psi = -\tfrac{1}{2} u_2 r_2^2 (1-\mu_2^2) + \text{constant,}$$

where $\mu = \cos\theta$.

Let us *assume*[‡] that

$$(v) \qquad \phi = \sum_{n=1}^{n=m} \left[A_n \frac{d^n}{ds_{12}^n}\left(\frac{1}{r_1}\right) + B_n \frac{d^n}{ds_{21}^n}\left(\frac{1}{r_2}\right) \right],$$

where m may be finite or infinite, A_n, B_n being arbitrary constants to be determined.

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[*] See Lamb's "*Hydrodynamics*," p 117. [†] Ibid, p. 122.

[‡] That this assumption is justifiable, will be clear from the actual values of the A's and the B's determined in the succeeding articles. The assumption was suggested by Prof. Pearson's paper.

The conditions (*i*) and (*ii*) are satisfied; it remains to determine the constants A_n and B_n so as to satisfy the conditions (*iii*) and (*iv*).

Now

$$r_2^2 = s^2 + r_1^2 - 2 s r_1 \mu_1 ;$$

$$\therefore \quad \frac{1}{r_2} = \frac{1}{s} \cdot \sum_0^\infty \left(\frac{r_1}{s}\right)^p P_p(\mu_1),$$

s being greater than r_1.

Again

$$\frac{d}{ds_{21}} = \frac{dr_1}{ds_{21}} \cdot \frac{d}{dr_1} + \frac{d\mu_1}{ds_{21}} \cdot \frac{d}{d\mu_1},$$

the subject to be operated upon being a function of r_1 and μ_1.

Now

$$ds_{21} \cdot \mu_1 = - dr_1,$$

$$ds_{21} \cdot \sin\theta_1 = r_1 \, d\theta_1 ;$$

therefore

$$\frac{d}{ds_{21}} = - \left(\frac{1-\mu_1^2}{r_1} \cdot \frac{d}{d\mu_1} + \mu_1 \frac{d}{dr_1}\right).$$

Hence

$$\frac{d}{ds_{21}} \left\{ r_1^p P_p(\mu_1) \right\} = - r_1^{p-1} \left\{ (1-\mu_1^2) \frac{dP_p(\mu_1)}{d\mu_1} + p\mu_1 P_p(\mu_1) \right\} = - p r_1^{p-1} P_{p-1}(\mu_1),$$

therefore

$$\frac{d^n}{ds_{21}^n} \left\{ r_1^p P_p(\mu_1) \right\} = (-1)^n \frac{p!}{(p-n)!} r_1^{p-n} P_{p-n}(\mu_1).$$

Hence

$$\frac{d^n}{ds_{21}^n} \left(\frac{1}{r_2}\right) = \frac{(-1)^n}{s^{n+1}} \sum_{p=n}^\infty \left(\frac{r_1}{s}\right)^{p-n} \frac{p!}{(p-n)!} P_{p-n}(\mu_1).$$

Further

$$\frac{d^n}{ds_{12}^n} \left(\frac{1}{r_1}\right) = \frac{(-1)^n n!}{r_1^{n+1}} P_n(\mu_1).$$

Hence

$$\phi = \sum_{n=1}^{n=m} \left[A_n \frac{(-1)^n n!}{r_1^{n+1}} P_n(\mu_1) + B_n \frac{(-1)^n}{s^{n+1}} \sum_{p=n}^\infty \frac{p!}{(p-n)!} \left(\frac{r_1}{s}\right)^{p-n} \times P_{p-n}(\mu_1) \right]$$

Re-arranging, we have

$$(vi)\quad \phi = \sum_{n=1}^{n=m} \left[\frac{(-1)^n n!}{r_1^{\,n+1}} \; P_n(\mu_1) \right]$$

$$- \sum_{k=0}^{\infty} \left[\frac{r_1^{\,k}}{s^{k+1}} \; R_{k+1} \, P_k(\mu_1) \right],$$

where

(vii)

$$R_{k+1} = (k+1)\frac{B_1}{s} - (k+2)(k+1)\frac{B_2}{s^2} + (k+3)(k+2)(k+1)\frac{B_3}{s^3} - \dots$$

Then*

(viii)

$$\psi = (1-\mu_1^{\,2}) \left[\sum_{n=1}^{n=m} \left\{ A_n \; \frac{(-1)^{n-1}(n-1)!}{r_1^{\,n}} \; P'_n \right\} - \sum_{k=1}^{\infty} \left\{ \frac{r_1^{\,k+1}}{s^{k+1}} \cdot \frac{R_{k+1}}{k+1} \; P'_k \right\} \right],$$

where P' is written for $\dfrac{d}{d\mu_1}\left\{ P(\mu_1) \right\}$.

§ 2.

The equations of the spheroids, with centres at O_1 and O_2, can be written as

$$r_1 = a\left\{1 + \epsilon_1 P_2(\mu_1)\right\},$$

(ix)

$$r_2 = b\left\{1 + \epsilon_2 P_2(\mu_2)\right\},$$

where ϵ_1 and ϵ_2 are two small quantities whose second and higher powers are supposed to be negligible; thus

$$r_1^{\,-n} = a^{-n}\left\{1 - n\,\epsilon_1 P_2(\mu_1)\right\},$$

$$r_1^{\,n+1} = a^{n+1}\left\{1 + (n+1)\,\epsilon_1 P_2(\mu_1)\right\}.$$

Then from (iii^1) and $(viii)$, we get

$$-\tfrac{1}{2} u_1 r_1^{\,2} (1-\mu_1^{\,2}) + \text{constant}$$

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* See Lamb's "*Hydrodynamics*," p. 120.

(x)

$$= (1-\mu_1^2)\left[\sum_{n=1}^{n=m}\left\{A_n\,\frac{(-1)^{n-1}(n-1)!}{r_1^n}\,P'_n\right\} - \sum_{k=1}^{\infty}\left\{\frac{r_1^{k+1}}{s^{k+1}}\cdot\frac{R_{k+1}}{k+1}\,P'_k\right\}\right],$$

when $r_1 = a\,\{1 + \epsilon_1\,P_2\,(\mu_1)\}$.

Evidently the constant is zero; we have therefore

$$-\tfrac{1}{2}\,u_1\,a^2\,\{1+2\,\epsilon_1\,P_2\}$$

$$= \sum_{n=1}^{n=m}\left[\frac{(-1)^{n-1}(n-1)!}{a^n}\,A_n\,\{1-n\epsilon_1\,P_2\}\,P'_n\right]$$

$$- \sum_{k=1}^{\infty}\left[\frac{a^{k+1}}{s^{k+1}}\cdot\frac{R_{k+1}}{k+1}\,\{1+(k+1)\,\epsilon_1\,P_2\}\,P'_k\right].$$

Since, by Christoffel's formula*

(xi)

$$P'_n = (2n-1)\,P_{n-1} + (2n-5)\,P_{n-3} + (2n-9)\,P_{n-5} + \ldots$$

and, consequently,

$$P_2\,P'_n = \frac{3n(n+1)}{2(2n+1)}\,(P_{n+1}-P_{n-1}) + (2n-1)\,P_{n-1} + (2n-5)\,P_{n-3} + \ldots,$$

the right-hand member in each equality ending with the term $3\,P_1$ if n is even and with the term P_0 if n is odd, we have

$$-\tfrac{1}{2}\,u_1\,a^2\,\{1+2\epsilon_1\,P_2\}$$

$$= \sum_{n=1}^{n=m}\frac{(-1)^{n-1}(n-1)!}{a^n}\,A_n\left[(2n-1)\,P_{n-1}+(2n-5)\,P_{n-3}+\ldots\right]$$

$$- \sum_{n=1}^{n=m}\frac{(-1)^{n-1}n!}{a^n}\,\epsilon_1\,A_n\left[\frac{3n(n+1)}{2(2n+1)}\,(P_{n+1}-P_{n-1})+(2n-1)\,P_{n-1}+\ldots\right]$$

(xii)

$$- \sum_{k=1}^{\infty}\frac{a^{k+1}}{s^{k+1}}\cdot\frac{R_{k+1}}{k+1}\left[(2k-1)\,P_{k-1}+(2k-5)\,P_{k-3}+\ldots\right]$$

$$- \sum_{k+1}^{\infty}\frac{a^{k+1}}{s^{k+1}}\cdot\epsilon_1\,R_{k+1}\left[\frac{3k(k+1)}{2(2k+1)}\,(P_{k+1}-P_{k-1})+(2k-1)\,P_{k-1}+\ldots\right].$$

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* See Heine's "*Kugelfunktionen*," Bd. I, p. 93.

It remains to equate to zero the co-efficients of the various zonal harmonics.

Equating the co-efficients of P_0,

$$-\tfrac{1}{2}u_1\,a^2 = \left[\frac{A_1}{a} + 2!\,\frac{A_3}{a^3} + 4!\,\frac{A_5}{a^5} + \ldots\right]$$

$$-\epsilon_1\left[3!\,\frac{A_3}{a^3} + 5!\,\frac{A_5}{a^5} + 7!\,\frac{A_7}{a^7} + \ldots\right]$$

(*xiii*)

$$-\left[\frac{a^2}{s^2}\cdot\frac{R_2}{2} + \frac{a^4}{s^4}\cdot\frac{R_4}{4} + \frac{a^6}{s^6}\cdot\frac{R_6}{6} + \ldots\right]$$

$$-\epsilon_1\left[\frac{a^4}{s^4}\,R_4 + \frac{a^6}{s^6}\,R_6 + \frac{a^8}{s^8}\,R_8 + \ldots\right].$$

Equating the co-efficients of P_2,

$$-u_1\,\epsilon_1\,a^2 = 5\left[2!\frac{A_3}{a^3} + 4!\frac{A_5}{a^5} + 6!\frac{A_7}{a^7} + \ldots\right]$$

$$-5\epsilon_1\left[3!\frac{A_3}{a^3} + 5!\frac{A_5}{a^5} + 7!\frac{A^7}{a^7} + \ldots\right]$$

(*xiv*)

$$-\epsilon_1\left[\frac{A_1}{a} - 3!\frac{18}{7}\cdot\frac{A_3}{a^3}\right]$$

$$-5\left[\frac{a^4}{s^4}\cdot\frac{R_4}{4} + \frac{a^6}{s^6}\cdot\frac{R_6}{6} + \ldots\right]$$

$$-5\epsilon_1\left[\frac{a^4}{s^4}\,R_4 + \frac{a^6}{s^6}\cdot R_6 + \ldots\right]$$

$$-\epsilon_1\left[\frac{a^2}{s^2}\,R_2 - \tfrac{18}{7}\cdot\frac{a^4}{s^4}\,R_4\right].$$

Subtracting (*xiv*) from (*xiii*) multiplied by 5, we get

$$-\tfrac{1}{2}u_1\,a^2\,(5-2\epsilon_1) = \frac{A_1}{a}\,(5+\epsilon_1) - \tfrac{108}{7}\,\epsilon_1\,\frac{A_3}{a^3}$$

$$-\tfrac{1}{2}\,\frac{a^2}{s^2}\,R_2\,(5-2\epsilon_1) - \tfrac{18}{7}\cdot\frac{a^4}{s^4}\,\epsilon_1\,R_4.$$

To evaluate A_1, we neglect the terms[*] involving A_3, B_2 and B_4. Then, up to the first approximation, we get

$$A_1 = -\tfrac{1}{2} u_1 a^3 \left(\frac{5-2\epsilon_1}{5+\epsilon_1}\right),$$

and consequently by symmetry,

$$B_1 = -\tfrac{1}{2} u_2 b^3 \left(\frac{5-2\epsilon_2}{5+\epsilon_2}\right).$$

It should be noted that, in getting these approximate values, we have neglected terms of the orders higher than $\left(\frac{\text{linear dimension}}{\text{central distance}}\right)^2$. Hence to this order of approximation, the velocity potential, for the motion of two spheroids in an infinite liquid, is given by

$$(xv)\quad \phi = -\tfrac{1}{2} u_1 a^3 \left(\frac{5-2\epsilon_1}{5+\epsilon_1}\right) \frac{d}{ds_{12}} \left(\frac{1}{r_1}\right) - \tfrac{1}{2} u_2 b^3 \left(\frac{5-2\epsilon_2}{5+\epsilon_2}\right) \frac{d}{ds_{21}} \left(\frac{1}{r_2}\right).$$

If we put each $\epsilon=0$, so that the spheroids become spheres of radii a and b, we have the velocity potential for the motion of two spheres in an infinite liquid, along their line of centres, to the first order of approximation, as

$$\phi = -\tfrac{1}{2} u_1 a^3 \frac{d}{ds_{12}} \left(\frac{1}{r_1}\right) - \tfrac{1}{2} u_2 b^3 \frac{d}{ds_{21}} \left(\frac{1}{r_2}\right),$$

which is Prof. Karl Pearson's result.[†]

§ 3.

Let us now proceed to approximate still closer to the value of ϕ by retaining the differential coefficients of the second and higher orders. Then from (*ii*), by equating the coefficients of P_n, we get, for $n > 0$ and $\neq 2$,

$$\begin{aligned} 0 = {} & (-1)^n (2n+1) \left[n! \frac{A_{n+1}}{a^{n+1}} + (n+2)! \frac{A_{n+3}}{a^{n+3}} + \dots \right] \\ & - (-1)^n (2n+1) \epsilon_1 \left[(n+1)! \frac{A_{n+1}}{a^{n+1}} + (n+3)! \frac{A_{n+3}}{a^{n+3}} + \dots \right] \\ & + (-1)^n \frac{A_{n+1}}{a^{n+1}} \epsilon_1 \frac{3(n+1)(n+2)}{2(2n+3)} (n+1)! \end{aligned}$$

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[*] The term involving A_3 is of the same order as ϵ_1^2; since A_3 is of the same order as ϵ_1, as will be clear from the value of A_3 obtained on p. 59. Therefore this term is negligible.

[†] Ibid p. 66.

$$- (-1)^{n-2} \frac{A_{n-1}}{a^{n-1}} \epsilon_1 \frac{3(n-1)n}{2(2n-1)} (n-1)!$$

$$- (2n+1) \left[\frac{a^{n+2}}{s^{n+2}} \cdot \frac{R_{n+2}}{n+2} + \frac{a^{n+4}}{s^{n+4}} \cdot \frac{R_{n+4}}{n+4} + \ldots \right]$$

$$- (2n+1) \epsilon_1 \left[\frac{a^{n+2}}{s^{n+2}} \cdot R_{n+2} + \frac{a^{n+4}}{s_{n+4}} R_{n+4} + \ldots \right]$$

$$+ \frac{a^{n+2}}{s^{n+2}} \epsilon_1 R_{n+2} \frac{3(n+1)(n+2)}{2(2n+3)}$$

$$- \frac{a^{n}}{s^{n}} \epsilon_1 R_{n} \frac{3(n-1)n}{2(2n-1)}.$$

Equating the coefficients of P_{n+2},

$$0 = (-1)^{n+2} (2n+5) \left[(n+2)! \frac{A_{n+3}}{a^{n+3}} + (n+4)! \frac{A_{n+5}}{a^{n+5}} + \ldots \right]$$

$$- (-1)^{n+2} (2n+5) \epsilon_1 \left[(n+3)! \frac{A_{n+3}}{a^{n+3}} + (n+5)! \frac{A_{n+5}}{a^{n+5}} + \ldots \right]$$

$$+ (-1)^{n+2} \frac{A_{n+3}}{a^{n+3}} \epsilon_1 \frac{3(n+3)(n+4)}{2(2n+7)} (n+3)!$$

$$- (-1)^{n} \frac{A_{n+1}}{a^{n+1}} \epsilon_1 \frac{3(n+1)(n+2)}{2(2n+3)} (n+1)!$$

$$- (2n+5) \left[\frac{a^{n+4}}{s^{n+4}} \cdot \frac{R_{n+4}}{n+4} + \frac{a^{n+6}}{s^{n+6}} \cdot \frac{R_{n+6}}{n+6} + \ldots \right].$$

$$- (2n+5)\epsilon_1 \left[\frac{a^{n+4}}{s^{n+1}} \cdot R_{n+4} + \frac{a^{n+6}}{s^{n+6}} R_{n+6} + \ldots \right]$$

$$+ \frac{a^{n+4}}{s^{n+4}} \epsilon_1 R_{n+4} \cdot \frac{3(n+3)(n+4)}{2(2n+7)}$$

$$- \frac{a^{n+2}}{s^{n+2}} \epsilon_1 R_{n+2} \frac{3(n+1)(n+2)}{2(2n+3)}.$$

Hence we get by subtraction

$$0 = (-1)^{n} n! \frac{A_{n+1}}{a^{n+1}} \left[(2n+1)(2n+5) - (n^2+3n-1)(n+1)\epsilon_1 \right]$$

$$- (-1)^{n} \epsilon_1 \frac{A_{n-1}}{a^{n-1}} \cdot \frac{3n(n-1)(2n+5)}{2(2n-1)} (n-1)\,!$$

$$- (-1)^{n+2} \epsilon_1 \frac{A_{n+3}}{a^{n+3}} \cdot \frac{3(n+3)(n+4)(2n+1)}{2(2n+7)} (n+3)\,!$$

$$- \frac{a^{n+2}}{s^{n+2}} \cdot \frac{R_{n+2}}{n+2} \Big[(2n+1)(2n+5) + (n^2+3n-1)(n+2)\epsilon_1 \Big]$$

$$- \frac{a^{n}}{s^{n}} \cdot R_n \epsilon_1 \frac{3(n-1)n(2n+5)}{2(2n-1)}$$

$$- \frac{a^{n+4}}{s^{n+4}} \epsilon_1 R_{n+4} \frac{3(n+3)(n+4)(2n+1)}{2(2n+7)} ;$$

of which the terms containing A_{n+3} and R_{n+4} may be neglected as small in comparison with the other terms.* Putting $n-2$ for n in this equation, we get

$$0 = (-1)^{n-2} (n-2)\,! \frac{A_{n-1}}{a^{n-1}} \Big[(2n-3)(2n+1) - (n^2-n-3)(n-1)\epsilon_1 \Big]$$

$$- (-1)^{n-2} \epsilon_1 \frac{A_{n-3}}{a^{n-3}} \cdot \frac{3(n-3)(n-2)(2n+1)}{2(2n-5)} (n-3)\,!$$

$$- \frac{a^{n}}{s^{n}} \cdot \frac{R_n}{n} \Big[(2n-3)(2n+1) + (n^2-n-3)\, n\, \epsilon_1 \Big]$$

$$- \frac{a^{n-2}}{s^{n-2}} \cdot \epsilon_1 R_{n-2} \frac{3(n-3)(n-2)(2n+1)}{2(2n-5)} .$$

Substituting the value of A_{n-1} from this equation in the previous equation, and remembering that $\epsilon_1{}^2$ is to be neglected, we get

$$0 = (-1)^{n} n\,! \frac{A_{n+1}}{a^{n+1}} \Big[(2n+1)(2n+5) - (n^2+3n-1)(n+1)\epsilon_1 \Big]$$

$$(xvi) \qquad - \frac{a^{n+2}}{s^{n+2}} \cdot \frac{R_{n+2}}{n+2} \Big[(2n+1)(2n+5) + (n^2+3n-1)(n+2)\epsilon_1 \Big]$$

$$- \frac{a^{n}}{s^{n}} \cdot R_n \epsilon_1 . \tfrac{3}{2}(n-1)(2n+5).$$

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* Putting $n+2$ for n in the above equation, we see at once that the term containing A_{n+3} is also of the same order as $\frac{a^{n+4}}{s^{n+4}} \epsilon_1 R_{n+4}$.

Hence from symmetry

$$O = (-1)^n\, n!\, \frac{B_{n+1}}{b^{n+1}} \left[(2n+1)(2n+5) - (n^2+3n-1)(n+1)\epsilon_2 \right]$$

$$\text{(xvii)} \qquad - \frac{b^{n+2}}{s^{n+2}} \cdot \frac{S_{n+2}}{n+2} \left[(2n+1)(2n+5) + (n^2+3n-1)(n+2)\epsilon_2 \right]$$

$$- \frac{b^n}{s^n} \cdot S_n\, \epsilon_2 .\, \frac{3(n-1)(2n+5)}{2}.$$

where

$$S_{k+1} = (k+1)\, \frac{A_1}{s} - (k+2)(k+1)\, \frac{A_2}{s^2} + \ \ldots$$

Thus we have two sets of equations from which by continued approximations we can determine the values of the A's and B's in terms of a, b and s.

§ 4.

Putting $n=4$ in the first equation of § 3, we get

$$O = 9 \left[4!\, \frac{A_5}{a^5} + 6!\, \frac{A_7}{a^7} + \quad \ldots \right]$$

$$- 9\, \epsilon_1 \left[5!\, \frac{A_5}{a^5} + 7!\, \frac{A_7}{a^7} + \quad \ldots \right]$$

$$+ \frac{A_5}{a^5} \quad \epsilon_1 .\, \frac{3\cdot5\cdot6}{2\cdot11}\, 5!$$

$$- \frac{A_3}{a^3} \quad \epsilon_1 \quad \frac{3\cdot3\cdot4}{2\cdot7}\, 3!$$

$$- 9 \left[\frac{a^6}{s^6} \cdot \frac{R_6}{6} + \frac{a^8}{s^8} \cdot \frac{R_8}{8} \quad + \quad \ldots \right]$$

$$- 9\epsilon_1 \left[\frac{a^6}{s^6} \cdot R_6 + \frac{a^8}{s^8}\, R_8 \quad + \quad \ldots \right]$$

$$+ \frac{a^6}{s^6}\, \epsilon_1\, R_6\, \frac{3\cdot5.6}{2\cdot11}$$

$$- \frac{a^4}{s^4}\, \epsilon_1\, R_4\, \frac{3\cdot3\cdot4}{2\cdot7}\, .$$

From this equation and (*xiv*) we get

$$-9u_1\epsilon_1 a^2 = \frac{A_3}{a^3}\,18(5-3\epsilon_1) - 9\epsilon_1\frac{A_1}{a} - 5\epsilon_1\frac{A_5}{a^5}\cdot\frac{3\cdot5\cdot6}{2\cdot11}\cdot 5\,!$$

$$-\frac{a^4}{s^4}\cdot\frac{R_4}{4}\,9(5+4\epsilon_1) - \frac{a^2}{s^2}\,9\epsilon_1 R_2 - 5\frac{a^6}{s^6}\epsilon_1 R_6\frac{3\cdot5\cdot6}{2\cdot11}$$

Neglecting the terms involving A_5, R_4 and R_6 which are of order higher than $\left(\frac{\text{linear dimension}}{\text{central distance}}\right)^4$, we get

$$-9u_1\epsilon_1 a^2 = \frac{A_3}{a^3}\,18(5-3\epsilon_1) - 9\epsilon_1\frac{A_1}{a} - 9\epsilon_1\frac{a^2}{s^2}R_2$$

Substituting the value of A_1, we get for the first approximation

$$A_3 = -\tfrac{3}{4}v_1 a^6\left(\frac{\epsilon_1}{5-3\epsilon_1}\right) - \frac{u_2}{2}\cdot\frac{a^5b^3}{s^3}\left(\frac{5-2\,\epsilon_2}{5+\epsilon_2}\right)\left(\frac{\epsilon_1}{5-3\epsilon_1}\right).$$

We next proceed to approximate to the remaining values of the A's and B's. We see that generally, for $n>o$ and $\neq 2$

$$A_{n+1} = \frac{(-1)^n}{n\,!}\cdot\frac{a^{2n+3}}{s^{n+2}}\cdot\frac{R_{n+2}}{n+2}$$

$$\left[1+\frac{(n^2+3n-1)(2n+3)\epsilon_1}{(2n+1)(2n+5)-(n^2+3n-1)(n+1)\epsilon_1}\right]$$

$$+(-1)^n\cdot\frac{a^{2n+1}}{s^n}\cdot R_n\epsilon_1\times$$

$$\frac{3(n-1)(2n+5))}{n\,!\,2[(2n+1)(2n+5)-(n^2+3n-1)(n+1)\epsilon_1]}\;;$$

and, consequently, by symmetry

$$B_{n+1} = \frac{(-1)^n}{n!}\cdot\frac{b^{2n+3}}{s^{n+2}}\cdot\frac{S_{n+2}}{n+2}$$

$$\left[1+\frac{(n^2+3n-1)(2n+3)\epsilon_2}{(2n+1)\;(2n+5)-(n^2+3n-1)\;(n+1)\epsilon_2}\right]$$

$$+(-1)^n\frac{b^{2n+1}}{s^n}\cdot S_n\epsilon_2\times\frac{3(n-1)(2n+5)}{n!2\,[(2n+1)(2n+5)-(n^2+3n-1)(n+1)\epsilon_2]}$$

Examples :—

$$A_2 = \frac{u_2}{2} \cdot \frac{a^6 b^3}{s^4} \left(\frac{5-2\epsilon_2}{5+\epsilon_2}\right) \left\{ 1 + \frac{5\epsilon_1}{7-2\epsilon_1} \right\},$$

$$A_4 = \frac{u_2}{2\cdot 3!} \cdot \frac{a^9 b^3}{s^6} \left(\frac{5-2\epsilon_2}{5+\epsilon_2}\right) \left\{ 1 + \frac{153\epsilon_1}{77-68\epsilon_1} \right\}$$

$$+ \frac{u_2}{2\cdot 2\,!} \cdot \frac{a^7 b^3}{s^4} \left(\frac{5-2\epsilon_2}{5+\epsilon_2}\right) \left(\frac{33\epsilon_1}{77-68\epsilon_1}\right),$$

$$B_2 = \frac{u_1}{2} \cdot \frac{b^6 a^3}{s^4} \left(\frac{5-2\epsilon_1}{5+\epsilon_1}\right) \left\{ 1 + \frac{5\epsilon_2}{7-2\epsilon_2} \right\},$$

$$B_4 = \frac{u_1}{2\cdot 3!} \quad \frac{b^9 a^3}{s^6} \left(\frac{5-2\epsilon_1}{5+\epsilon_1}\right) \left\{ 1 + \frac{153\epsilon_2}{77-68\epsilon_2} \right\}$$

$$- \frac{u_1}{2\cdot 2!} \cdot \frac{b^7 a^3}{s^4} \left(\frac{5-2\epsilon_1}{5+\epsilon_1}\right) \left(\frac{33\epsilon_2}{77-68\epsilon_2}\right);$$